

# Poisson-Nernst-Planck Systems for Narrow Tubular-like Membrane Channels

Weishi Liu\*

Department of Mathematics  
University of Kansas, Lawrence, KS 66045

Bixiang Wang†

Department of Mathematics  
New Mexico Institute of Mining and Technology, Socorro, NM 87801

## Abstract

We study global dynamics of the Poisson-Nernst-Planck (PNP) system for flows of two types of ions through a narrow tubular-like membrane channel. As the radius of the cross-section of the three-dimensional tubular-like membrane channel approaches zero, a one-dimensional limiting PNP system is derived. This one-dimensional limiting system differs from previous studied one-dimensional PNP systems in that it encodes the defining geometry of the three-dimensional membrane channel. To justify this limiting process, we show that the global attractors of the three-dimensional PNP systems are upper semi-continuous to that of the limiting PNP system. We then examine the dynamics of the one-dimensional limiting PNP system. For large Debye number, the steady-state of the one-dimensional limiting PNP system is completely analyzed using the geometric singular perturbation theory. For a special case, an entropy-type Lyapunov functional is constructed to show the global, asymptotic stability of the steady-state.

**Key words.** Poisson-Nernst-Planck system, global attractor, steady solution.

**MSC 2000.** Primary 37L55. Secondary 35B40.

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\*Partially supported by NSF Grant DMS-0406998.

†Partially supported by NSF Grant DMS-0703521.

# 1 Introduction

Poisson-Nernst-Planck (PNP) systems serve as basic electro-diffusion equations modeling, for example, ion flow through membrane channels, transport of holes and electrons in semiconductors (see, e.g., [1, 2, 21, 27] and the references therein). There are many excellent works on derivation of PNP systems from Boltzmann equations (see [1] and reference therein) by assuming the collision time is much smaller than the characteristic time. The PNP systems have been studied under various physical relevant boundary conditions such as the non-flux, homogeneous Dirichlet and Neumann boundary conditions. For those types of boundary conditions, in addition to the total charge conservation and the existence of various first integrals, Boltzmann H-functionals or entropy-like functionals are successfully constructed, which, together with the advances of Csiszár-Kullback-type or logarithmic Sobolev inequalities, are applied to investigate the asymptotic behavior of the PNP systems and stability of steady-state or self-similar solutions (see, e.g., [3, 4, 8, 9, 10, 24, 28]). In the context of ion flow through membrane channels, it is physically unreasonable to impose the above mentioned boundary conditions on the whole boundary, particularly at the two “ends” of the channels. Instead, non-homogeneous Dirichlet conditions on the two “ends” are typically assumed. PNP systems supplemented with this type of boundary conditions result in quite different dynamical behavior. The total charges within the channels are not conserved and entropy-like functionals are not available in general, and, most significantly, the asymptotic behavior is different as showed in special cases in the present paper.

In this work, we start a systematic study of the PNP systems modeling ion flows through narrow tubular-like membrane channels in physiology. For definiteness, consider flow of two types of ions,  $S_1$  and  $S_2$ , one with the positive valence  $\alpha_1 > 0$  and the other with the negative valence  $-\alpha_2 < 0$ , passing through a membrane channel with length normalized from  $X = 0$  to  $X = 1$ . Denote the concentrations of  $S_1$  and  $S_2$  at location  $(X, Y, Z)$  and time  $t$  by  $c_1(t, X, Y, Z)$  and  $c_2(t, X, Y, Z)$ . Then the electric potential  $\Phi(t, X, Y, Z)$  in the channel is governed by the Poisson equation

$$\Delta \Phi = -\lambda(\alpha_1 c_1 - \alpha_2 c_2),$$

where the parameter  $\lambda$  is the Debye number related to the ratio of the Debye length to a characteristic length scale. The flux densities,  $\bar{J}_1$  and  $\bar{J}_2$ , of the two ions contributed from the concentration gradients of the two ions and the electric field satisfy the Nernst-Planck equations

$$D_1(\nabla c_1 + \alpha_1 c_1 \nabla \Phi) = -\bar{J}_1 \quad \text{and} \quad D_2(\nabla c_2 - \alpha_2 c_2 \nabla \Phi) = -\bar{J}_2,$$

and the continuity equations

$$\frac{\partial c_1}{\partial t} + \nabla \bar{J}_1 = 0, \quad \frac{\partial c_2}{\partial t} + \nabla \bar{J}_2 = 0,$$

where  $D_1$  and  $D_2$  are the diffusion constants of ions  $S_1$  and  $S_2$  relative to the membrane channel. The Poisson-Nernst-Planck system is thus given by

$$\begin{aligned}\Delta\Phi &= -\lambda(\alpha_1 c_1 - \alpha_2 c_2), \\ \frac{\partial c_1}{\partial t} &= D_1 \nabla \cdot (\nabla c_1 + \alpha_1 c_1 \nabla \Phi), \\ \frac{\partial c_2}{\partial t} &= D_2 \nabla \cdot (\nabla c_2 - \alpha_2 c_2 \nabla \Phi).\end{aligned}\tag{1.1}$$

PNP systems have been studied by many authors (see, e.g., [1, 2, 3, 4, 8, 9, 10, 15, 16, 17, 21, 24, 25, 27, 28]). Many works have been attributed to the one-dimensional PNP systems and particularly the steady-state problems (see, e.g., [15, 16, 25, 17]). Consideration of one-dimensional PNP systems is motivated naturally by the fact that membrane channels are narrow. To make this reduction more rigorous, we present a mathematical framework based on the ideas in [12, 13] and investigate mathematically the limiting process as the three-dimensional domain shrinks to a line segment. More specifically, starting with the situation that  $\Omega_\epsilon$  is a revolution domain about its length ( $\epsilon$  is related to the maximal radius of the cross-sections of the channel), we will derive a one-dimensional limiting PNP system as  $\epsilon \rightarrow 0$ . Differing from the simple one-dimensional version of the PNP system, this limiting PNP system encodes the defining geometry of the three-dimensional channel. As the first step in justifying the limiting process, we show the upper semi-continuity of the global attractors  $\mathcal{A}_\epsilon$  of the three-dimensional systems at  $\epsilon = 0$ . The existence of global attractors for the PNP systems can be found in [9]. It is expected that if the one-dimensional limiting system is structurally stable, then its dynamics determines that of three-dimensional system for  $\epsilon > 0$  small. We will thus examine, in this paper, the steady-state problem of the one-dimensional limiting system. For large Debye number, the steady-state problem can be viewed as a singularly perturbed one. We show that this problem can be completely analyzed using the geometric singular perturbation theory as in [23].

The rest of the paper is organized as follows. In Section 2, we give detailed formulation of our problem. The domain for the three-dimensional PNP system will be specified and, as the domain shrinks to a one-dimensional segment, a one-dimensional limiting PNP system is derived. We then state our results on the upper semi-continuity of attractors, on the singular boundary value problem of the steady-state PNP system. The proofs are provided in Sections 3 and 4. In Section 3, after some technical preparations, we show the upper semi-continuity of attractors. Section 4 is devoted to the geometric analysis of the singularly perturbed steady-state problem of the one-dimensional limiting PNP system. At the end of this section, a special case is studied for which a  $H$ -function is found and used to establish the asymptotic stability of the steady-state.

## 2 Formulation of the problem and the statements of main results

### 2.1 Three-dimensional PNP and a one-dimensional limit

We start with setting up our problem. The membrane channel considered here is special and will be viewed as a tubular-like domain  $\Omega_\epsilon$  in  $\mathbb{R}^3$  as follows:

$$\Omega_\epsilon = \{(X, Y, Z) : 0 < X < 1, Y^2 + Z^2 < g^2(X, \epsilon)\},$$

where  $g$  is a smooth (at least  $\mathcal{C}^3$ ) function satisfying

$$g(X, 0) = 0 \text{ and } g_0(X) = \frac{\partial g}{\partial \epsilon}(X, 0) > 0 \text{ for } X \in [0, 1]. \quad (2.1)$$

The positive parameter  $\epsilon$  measures the sizes of cross-sections of the membrane channel. For a technical reason (used in Lemma 3.1), we also assume that

$$\frac{\partial g}{\partial X}(0, \epsilon) = \frac{\partial g}{\partial X}(1, \epsilon) = 0.$$

The boundary  $\partial\Omega_\epsilon$  of  $\Omega_\epsilon$  will be divided into three portions as follows:

$$\begin{aligned} \hat{L}_\epsilon &= \{(X, Y, Z) \in \partial\Omega_\epsilon : X = 0\}, \\ \hat{R}_\epsilon &= \{(X, Y, Z) \in \partial\Omega_\epsilon : X = 1\}, \\ \hat{M}_\epsilon &= \{(X, Y, Z) \in \partial\Omega_\epsilon : Y^2 + Z^2 = g^2(X, \epsilon)\}. \end{aligned}$$

Thus,  $\hat{L}_\epsilon$  and  $\hat{R}_\epsilon$  are viewed as the two ends of the channel and  $\hat{M}_\epsilon$  the wall of the channel. The boundary conditions considered in this paper are

$$\begin{aligned} \Phi|_{\hat{L}_\epsilon} &= \phi_0 > 0, \quad \Phi|_{\hat{R}_\epsilon} = 0, \quad c_k|_{\hat{L}_\epsilon} = l_k > 0, \quad c_k|_{\hat{R}_\epsilon} = r_k > 0, \\ \frac{\partial \Phi}{\partial \mathbf{n}}|_{\hat{M}_\epsilon} &= \frac{\partial c_k}{\partial \mathbf{n}}|_{\hat{M}_\epsilon} = 0, \end{aligned} \quad (2.2)$$

where  $\phi_0$ ,  $l_k$  and  $r_k$  ( $k = 1, 2$ ) are constants, and  $\mathbf{n}$  is the outward unit normal vector to  $\hat{M}_\epsilon$ . Although the most natural boundary conditions on  $\hat{M}_\epsilon$  would be the non-flux one

$$\left( \frac{\partial c_1}{\partial \mathbf{n}} + \alpha_1 c_1 \frac{\partial \Phi}{\partial \mathbf{n}} \right)|_{\hat{M}_\epsilon} = \left( \frac{\partial c_2}{\partial \mathbf{n}} - \alpha_2 c_2 \frac{\partial \Phi}{\partial \mathbf{n}} \right)|_{\hat{M}_\epsilon} = 0,$$

the above homogeneous Neumann conditions on  $\hat{M}_\epsilon$  are reasonable (they are the consequences of the non-flux and zero-outward electric field conditions).

In this paper, we are interested in the limiting behavior of the PNP system when the three-dimensional tubular-like domain  $\Omega_\epsilon$  collapses to a one-dimensional interval as  $\epsilon \rightarrow 0$ . Naturally we expect a one-dimensional limiting system whose global dynamics is comparable with those of PNP

systems for  $\epsilon > 0$  small. This important idea was applied by many researchers in studying the dynamics of equations defined on thin domains (see, e.g., [12, 13, 26, 29]). We follow the procedure in [13] to derive a one-dimensional limiting system but avoid expressing differential operators and transformations in local coordinates. As a result, the expected one-dimensional limiting system is more transparent.

To derive the limiting PNP system, we transfer the  $\epsilon$ -dependent domain  $\Omega_\epsilon$  into a fixed domain  $\Omega = [0, 1] \times \mathbb{D}$ , where  $\mathbb{D}$  is the unit disk, by applying the following change of coordinates:

$$x = X, \quad y = \frac{Y}{g(X, \epsilon)}, \quad z = \frac{Z}{g(X, \epsilon)}. \quad (2.3)$$

In the sequel, we denote by  $L$ ,  $R$  and  $M$ , respectively, the boundaries of  $\Omega$  corresponding to  $\hat{L}_\epsilon$ ,  $\hat{R}_\epsilon$  and  $\hat{M}_\epsilon$  under the transformation. Let  $J$  denote the Jacobian matrix of the change of coordinates. Then,

$$J = \frac{\partial(x, y, z)}{\partial(X, Y, Z)} = \frac{1}{g^2} \begin{pmatrix} g^2 & 0 & 0 \\ -gg_x y & g & 0 \\ -gg_x z & 0 & g \end{pmatrix}, \quad J^{-1} = \frac{\partial(X, Y, Z)}{\partial(x, y, z)} = \begin{pmatrix} 1 & 0 & 0 \\ g_x y & g & 0 \\ g_x z & 0 & g \end{pmatrix}$$

with  $\det(J^{-1}) = g^2(x, \epsilon)$ , and

$$JJ^T = \frac{1}{g^4} \begin{pmatrix} g^4 & -g^3 g_x y & -g^3 g_x z \\ -g^3 g_x y & g^2 + g^2 g_x^2 y^2 & g^2 g_x^2 y z \\ -g^3 g_x z & g^2 g_x^2 y z & g^2 + g^2 g_x^2 z^2 \end{pmatrix}.$$

The following result, which can be verified by direct computations, is useful for a clean derivation of a limiting PNP system.

**Lemma 2.1.** *Let  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\psi(p) = q$ , be a diffeomorphism, and let  $J(q) = \frac{\partial q}{\partial p}(\psi^{-1}(q))$  be the Jacobian matrix and  $d(q) = (\det J(q))^{-1}$ . If  $\alpha(p) = \beta(\psi(p)) : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth function, then the gradients in the two coordinates are related as*

$$\nabla_p \alpha(p) = J^T(q) \nabla_q \beta(q).$$

Further, if  $\sum_{j=1}^n \frac{\partial}{\partial q_j} \left( d(q) \frac{\partial q_j}{\partial p_i} \right) = 0$  for all  $i = 1, \dots, n$ , and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a smooth vector field, then  $F(p) = f(\psi(p))$  satisfies

$$\nabla_p \cdot F(p) = \frac{1}{d(q)} \nabla_q \cdot (d(q) J(q) f(q)),$$

and hence, the Laplace operators are related as

$$\Delta_p \alpha(p) = \frac{1}{d(q)} \nabla_q \cdot (d(q) J(q) J^T(q) \nabla_q \beta(q)).$$

It can be checked that the change of variables in (2.3) with  $p = (X, Y, Z)$  and  $q = (x, y, z)$  satisfies

$$\sum_{j=1}^n \frac{\partial}{\partial q_j} \left( d(q) \frac{\partial q_j}{\partial p_i} \right) = 0.$$

Therefore, applying Lemma 2.1, system (1.1) can be rewritten, in terms of  $(x, y, z)$ , as follows.

$$\begin{aligned} \frac{1}{g^2} \nabla \cdot (g^2 J J^T \nabla \Phi) &= -\lambda(\alpha_1 c_1 - \alpha_2 c_2), \\ \frac{\partial c_1}{\partial t} &= \frac{D_1}{g^2} \nabla \cdot (g^2 J J^T \nabla c_1 + \alpha_1 c_1 g^2 J J^T \nabla \Phi), \\ \frac{\partial c_2}{\partial t} &= \frac{D_2}{g^2} \nabla \cdot (g^2 J J^T \nabla c_2 - \alpha_2 c_2 g^2 J J^T \nabla \Phi), \end{aligned} \quad (2.4)$$

with the boundary conditions

$$\begin{aligned} \Phi|_L &= \phi_0, \quad \Phi|_R = 0, \quad c_k|_L = l_k, \quad c_k|_R = r_k, \\ \langle \nabla \Phi, J J^T \nu \rangle|_M &= \langle \nabla c_k, J J^T \nu \rangle|_M = 0, \end{aligned} \quad (2.5)$$

where  $k = 1, 2$  and  $\nu$  is the outward unit normal vector to  $M$ .

By inspecting the structural dependence of  $J J^T$  on  $\epsilon$ , we expect the one-dimensional limiting PNP system to be

$$\begin{aligned} \frac{1}{g_0^2} \frac{\partial}{\partial x} \left( g_0^2 \frac{\partial}{\partial x} \Phi \right) &= -\lambda(\alpha_1 c_1 - \alpha_2 c_2), \\ \frac{\partial c_1}{\partial t} &= \frac{D_1}{g_0^2} \frac{\partial}{\partial x} \left( g_0^2 \frac{\partial}{\partial x} c_1 + \alpha_1 c_1 g_0^2 \frac{\partial}{\partial x} \Phi \right), \\ \frac{\partial c_2}{\partial t} &= \frac{D_2}{g_0^2} \frac{\partial}{\partial x} \left( g_0^2 \frac{\partial}{\partial x} c_2 - \alpha_2 c_2 g_0^2 \frac{\partial}{\partial x} \Phi \right), \end{aligned} \quad (2.6)$$

on  $x \in (0, 1)$  with the boundary conditions

$$\Phi(t, 0) = \phi_0, \quad \Phi(t, 1) = 0, \quad c_k(t, 0) = l_k, \quad c_k(t, 1) = r_k, \quad (2.7)$$

where  $g_0(x)$  is defined in (2.1).

It was shown in [9] that, for any  $\epsilon > 0$ , the three-dimensional PNP system has a global attractor  $\mathcal{A}_\epsilon$  which is a compact subset and attracts all solutions with respect to the norm topology of  $H^1 \times H^1$ . This result is based on an invariant principle discovered in [10, 11, 9, 28] for the van Roosbroeck models of semi-conductor. The PNP systems are basically the same as the van Roosbroeck models and we recall the invariant principle using the above setting.

**Proposition 2.2.** *Let  $M$  be a positive constant with*

$$M \geq \max\{\alpha_1 l_1, \alpha_1 r_1, \alpha_2 l_2, \alpha_2 r_2\},$$

and let  $\tilde{\Sigma}$  be the subset of  $H^1(\Omega_\epsilon) \times H^1(\Omega_\epsilon)$  given by

$$\tilde{\Sigma} = \{(c_1, c_2) \in H^1(\Omega_\epsilon) \times H^1(\Omega_\epsilon) : 0 \leq \alpha_1 c_1 \leq M, 0 \leq \alpha_2 c_2 \leq M\}.$$

Then  $\tilde{\Sigma}$  is positively invariant for the PNP system. More precisely, if the initial datum  $(c_1(0), c_2(0)) \in \tilde{\Sigma}$  and  $(c_1, c_2)$  is the solution of the PNP system, then  $(c_1(t), c_2(t)) \in \tilde{\Sigma}$  for all  $t \geq 0$ .

We remark that, for PNP systems with three or more types of ions, the above invariant principle is not available. It is not clear to us whether or not a similar principle still holds in this case. PNP systems with more than two types of ions are worth further studying.

The results in [10, 11, 9, 28] show also that the one-dimensional problem (2.6)-(2.7) has a positively invariant region

$$\tilde{\Sigma}_0 = \{(c_1, c_2) \in H^1(0, 1) \times H^1(0, 1) : 0 \leq \alpha_1 c_1 \leq M, 0 \leq \alpha_2 c_2 \leq M\},$$

where  $M$  is the constant in Proposition 2.2, and problem (2.6)-(2.7) is globally well-posed in  $\tilde{\Sigma}_0$  and has a global attractor  $\mathcal{A}_0$  in  $\tilde{\Sigma}_0 \cap H^1(0, 1) \times H^1(0, 1)$ .

Our first result claims that the global attractors  $\mathcal{A}_\epsilon$  of the three-dimensional PNP systems are upper semi-continuous to the global attractor  $\mathcal{A}_0$  of the one-dimensional limiting system as  $\epsilon \rightarrow 0$ , which partially justify the limiting process.

**Theorem 2.3.** *The global attractors  $\mathcal{A}_\epsilon$  of the three-dimensional PNP systems are upper semi-continuous at  $\epsilon = 0$ , that is, for any  $\eta > 0$ , there exists a positive number  $\epsilon_1 = \epsilon_1(\eta)$  such that for all  $0 < \epsilon \leq \epsilon_1$  and all  $w \in \mathcal{A}_\epsilon$ ,*

$$\text{dist}_{X_\epsilon}(w, \mathcal{A}_0) \leq \eta,$$

where  $X_\epsilon = \{w : \|w\|_{X_\epsilon}^2 = \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \frac{1}{\epsilon^2} \|w_y\|_{L^2}^2 + \frac{1}{\epsilon^2} \|w_z\|_{L^2}^2 < \infty\}$ .

## 2.2 Steady-state problem of the one-dimensional limiting PNP system

The steady-state of problem (2.6) and (2.7) can be rewritten as

$$\begin{aligned} \mu^2 \frac{d}{dx} \left( h(x) \frac{d\phi}{dx} \right) &= -h(x)(\alpha c_1 - \beta c_2), \quad \frac{dJ_1}{dx} = 0, \quad \frac{dJ_2}{dx} = 0, \\ h(x) \frac{dc_1}{dx} + \alpha c_1 h(x) \frac{d\phi}{dx} &= -J_1, \quad h(x) \frac{dc_2}{dx} - \beta c_2 h(x) \frac{d\phi}{dx} = -J_2 \end{aligned} \tag{2.8}$$

where  $\mu^2 = 1/\lambda$ ,  $J_1 = \bar{J}_1/D_1$ ,  $J_2 = \bar{J}_2/D_2$  and  $h(x) = g_0^2(x)$ , and the boundary conditions are

$$\begin{aligned} \phi(0) &= \phi_0, \quad c_1(0) = l_1, \quad c_2(0) = l_2, \\ \phi(1) &= 0, \quad c_1(1) = r_1, \quad c_2(1) = r_2. \end{aligned} \tag{2.9}$$

Since  $\lambda$  is large, we can treat the problem (2.8) and (2.9) as a singularly perturbed problem with  $\mu$  as the singular parameter. We will recast the singularly perturbed PNP system into a system of first order equations.

Denote derivatives with respect to  $x$  by overdot and introduce

$$\tau = x, \quad u = \mu h(\tau) \dot{\phi}, \quad v = -h(\tau)(\alpha_1 c_1 - \alpha_2 c_2), \quad \text{and} \quad w = \alpha_1^2 c_1 + \alpha_2^2 c_2.$$

System (2.8) becomes

$$\begin{aligned} \mu \dot{\phi} &= \frac{1}{h(\tau)} u, \quad \mu \dot{u} = v, \quad \mu \dot{v} = uw + \mu \frac{h_\tau(\tau)}{h(\tau)} v + \mu(\alpha_1 J_1 - \alpha_2 J_2), \\ \mu \dot{w} &= \frac{\alpha_1 \alpha_2}{h^2(\tau)} uv + \frac{\alpha_2 - \alpha_1}{h(\tau)} uw - \frac{\mu}{h(\tau)} (\alpha_1^2 J_1 + \alpha_2^2 J_2), \\ \dot{J}_1 &= 0, \quad \dot{J}_2 = 0, \quad \dot{\tau} = 1. \end{aligned} \tag{2.10}$$

System (2.10) – *the slow system* – will be treated as a dynamical system with the phase space

$$\mathbb{R}^7 = \{(\phi, u, v, w, J_1, J_2, \tau)\}$$

and the independent variable  $x$  will be viewed as time. The boundary condition (2.9) becomes

$$\begin{aligned} \phi(0) &= \phi_0, \quad v(0) = -h(0)(\alpha_1 l_1 - \alpha_2 l_2), \quad w(0) = \alpha^2 L_1 + \beta^2 L_2, \quad \tau(0) = 0, \\ \phi(1) &= 0, \quad v(1) = -h(1)(\alpha_1 r_1 - \alpha_2 r_2), \quad w(1) = \alpha_1^2 r_1 + \alpha_2^2 r_2, \quad \tau(1) = 1. \end{aligned} \tag{2.11}$$

Setting  $\mu = 0$  in system (2.10), we get the limiting slow system

$$\begin{aligned} 0 &= \frac{1}{h(\tau)} u, \quad 0 = v, \quad 0 = uw, \\ 0 &= \frac{\alpha_1 \alpha_2}{h^2(\tau)} uv + \frac{\alpha_2 - \alpha_1}{h(\tau)} uw, \\ \dot{J}_1 &= 0, \quad \dot{J}_2 = 0, \quad \dot{\tau} = 1. \end{aligned} \tag{2.12}$$

The set  $\mathcal{Z}_0 = \{u = v = 0\}$  is called *the slow manifold* which supports the regular layer of the boundary value problem. The regular layer will not satisfy all conditions in (2.11) if  $\alpha_2 l_2 - \alpha_1 l_1 \neq 0$  or  $\alpha_2 r_2 - \alpha_1 r_1 \neq 0$ , and this defect has to be remedied by boundary layers. The boundary layer behavior will be determined by the fast system resulting from the slow system (2.10) by the rescaling of time  $x = \mu \xi$ . Thus, in terms of  $\xi$ , *the fast system* of (2.10) is

$$\begin{aligned} \phi' &= \frac{1}{h(\tau)} u, \quad u' = v, \quad v' = uw + \mu \frac{h_\tau(\tau)}{h(\tau)} v + \mu(\alpha_1 J_1 - \alpha_2 J_2), \\ w' &= \frac{\alpha_1 \alpha_2}{h^2(\tau)} uv + \frac{\alpha_2 - \alpha_1}{h(\tau)} uw - \frac{\mu}{h(\tau)} (\alpha_1^2 J_1 + \alpha_2^2 J_2), \\ J_1' &= 0, \quad J_2' = 0, \quad \tau' = \mu. \end{aligned} \tag{2.13}$$



where prime denotes the derivative with respect to the variable  $\xi$ . The limiting fast system at  $\mu = 0$  is

$$\begin{aligned} \phi' &= \frac{1}{h(\tau)}u, \quad u' = v, \quad v' = uw, \quad w' = \frac{\alpha_1\alpha_2}{h^2(\tau)}uv + \frac{\alpha_2 - \alpha_1}{h(\tau)}uw, \\ J_1' &= 0, \quad J_2' = 0, \quad \tau' = 0. \end{aligned} \quad (2.14)$$

The slow manifold  $\mathcal{Z}_0$  is precisely the set of equilibria of (2.14).

Concerning the steady-state problem of the one-dimensional limiting PNP system, we have

**Theorem 2.4.** *Assume that  $\alpha_1 l_1 \neq \alpha_2 l_2$  and  $\alpha_1 r_1 \neq \alpha_2 r_2$  (otherwise, see Remark 4.1). For  $\mu > 0$  small, the boundary value problem (2.10) and (2.11) has a unique solution near a singular orbit. The singular orbit is the union of two fast orbits of system (2.14) representing the boundary layers and one slow orbit of a blow-up system of (2.12) (see Section 4 for details) for the regular layer. The limiting flux densities are explicitly given by*

$$\begin{aligned} \bar{J}_1 = D_1 J_1 &= \frac{D_1 \left( \ln \frac{r_1}{l_1} - \alpha_1 \phi_0 \right) \left( (\alpha_1 l_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 l_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} - (\alpha_1 r_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 r_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} \right)}{\left( \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} \ln \frac{r_1}{l_1} + \frac{\alpha_1^2}{\alpha_1 + \alpha_2} \ln \frac{r_2}{l_2} \right) \int_0^1 h^{-1}(x) dx}, \\ \bar{J}_2 = D_2 J_2 &= \frac{D_2 \left( \ln \frac{r_2}{l_2} + \alpha_2 \phi_0 \right) \left( (\alpha_1 l_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 l_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} - (\alpha_1 r_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 r_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} \right)}{\left( \frac{\alpha_2^2}{\alpha_1 + \alpha_2} \ln \frac{r_1}{l_1} + \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} \ln \frac{r_2}{l_2} \right) \int_0^1 h^{-1}(x) dx}. \end{aligned}$$

*Remark 2.1.* Note that the factor  $\int_0^1 h^{-1}(x) dx$  on the denominators in the expressions for the flux densities  $J_1$  and  $J_2$  reflects the effect of the geometry of the three-dimensional channel. Let's compare this effect with that of a cylindrical channel where the wall is defined by  $\{Y^2 + Z^2 = \epsilon\}$ . In this case, the corresponding integral factor on the denominators for the flux densities  $J_1$  and  $J_2$  is 1. The volume of the channel is  $\pi\epsilon^2$ . For general channels that we considered here, we thus assume

$$\text{Vol} = \int_0^1 \pi g^2(x; \epsilon) dx = \pi\epsilon^2.$$

From which we have  $\int_0^1 h(x) dx = 1$ . Therefore,

$$1 = \left( \int_0^1 h^{-1/2}(x) h^{1/2}(x) dx \right)^2 \leq \int_0^1 h^{-1}(x) dx \int_0^1 h(x) dx = \int_0^1 h^{-1}(x) dx.$$

The inequality indicates that *the more complicated the geometry of the channel, the smaller the flux for the ion flow*, which agrees with our common sense.

### 3 Upper semi-continuity of attractors

#### 3.1 Homogenization of boundary conditions

In this section, we convert the non-homogeneous Dirichlet boundary conditions on  $L \cup R$  in (2.5) to homogeneous ones, while keeping the homogeneous Neumann boundary conditions on  $M$ . For this purpose, the following technical result is needed.

**Lemma 3.1.** *Let  $h : [0, 1] \rightarrow \mathbb{R}$  be a smooth function. Then, for any  $\epsilon > 0$ , there is a function  $H^\epsilon : \Omega_\epsilon \rightarrow \mathbb{R}$  such that  $H^\epsilon(X, 0, 0) = h(X)$ ,  $H^\epsilon(0, Y, Z) = h(0)$ ,  $H^\epsilon(1, Y, Z) = h(1)$ , and  $\langle \nabla H^\epsilon(X, Y, Z), \mathbf{n} \rangle = 0$  for  $(X, Y, Z) \in \hat{M}_\epsilon$ .*

*Proof.* We provide a specific construction of a function  $H^\epsilon$ . For convenience, hereafter, we denote by  $g'(X, \epsilon) = \frac{\partial g}{\partial X}(X, \epsilon)$ . For any  $\epsilon > 0$  and  $X_0 \in [0, 1]$ , let  $X = \psi^\epsilon(t, X_0)$  be the solution of

$$\frac{dX}{dt} = -t \frac{g'(X, \epsilon)}{g(X, \epsilon)} \quad (3.1)$$

with  $\psi^\epsilon(0, X_0) = X_0$ . It is easy to see that  $\psi^\epsilon(t, X_0)$  is even in  $t$  from the equation. Since  $g'(0, \epsilon) = g'(1, \epsilon) = 0$ ,  $\psi^\epsilon(t, 0) = 0$  and  $\psi^\epsilon(t, 1) = 1$  for all  $t$ . Therefore, for any  $(X, t) \in [0, 1] \times [0, g(X, \epsilon)]$ , there is a unique  $X_0 \in [0, 1]$  such that  $X = \psi^\epsilon(t, X_0)$ , and hence, for any  $(X, Y, Z) \in \Omega_\epsilon$ , there is a unique  $X_0 \in [0, 1]$  such that  $X = \psi^\epsilon(\sqrt{Y^2 + Z^2}, X_0)$ . Set  $H^\epsilon(X, Y, Z) = h(X_0)$  if  $X = \psi^\epsilon(\sqrt{Y^2 + Z^2}, X_0)$ . Then,  $H^\epsilon(X, 0, 0) = h(X)$ ,  $H^\epsilon(0, Y, Z) = h(0)$  and  $H^\epsilon(1, Y, Z) = h(1)$ . It remains to show that, for  $(X, Y, Z) \in \hat{M}_\epsilon$ ,  $\langle \nabla H^\epsilon(X, Y, Z), \mathbf{n} \rangle = 0$ . For any  $X_0 \in [0, 1]$ , the set

$$D(X_0) = \{(X, Y, Z) : X = \psi^\epsilon(\sqrt{Y^2 + Z^2}, X_0)\} = \{(X, Y, Z) : H(X, Y, Z) = h(X_0)\},$$

is a level set of  $H^\epsilon$ . Note also that the curve  $\{(X, Y, 0) : X = \psi^\epsilon(Y, X_0)\}$  lies on  $D(X_0)$  and it is a solution curve to (3.1) if  $Y$  is viewed as the  $t$ -variable. Therefore, at  $(X, Y, 0) = (X, g(X, \epsilon), 0) \in D(X_0) \cap \hat{M}_\epsilon$ , the vector

$$\left(-Y \frac{g'(X, \epsilon)}{g(X, \epsilon)}, 1, 0\right) = (-g'(X, \epsilon), 1, 0)$$

is tangent to  $D(X_0)$ , and hence,  $\langle \nabla H^\epsilon(X, g(X, \epsilon), 0), (-g'(X, \epsilon), 1, 0) \rangle = 0$ . Since  $\mathbf{n}$  is parallel to  $(-g'(X, \epsilon), 1, 0)$ ,  $\langle \nabla H^\epsilon(X, g(X, \epsilon), 0), \mathbf{n} \rangle = 0$ . Due to the rotation symmetry of  $\hat{M}_\epsilon$  and  $H^\epsilon$  about the  $X$ -axis, we conclude that, for  $(X, Y, Z) \in \hat{M}_\epsilon$ ,  $\langle \nabla H^\epsilon(X, Y, Z), \mathbf{n} \rangle = 0$ .  $\square$

Let  $L_k^0(X)$ , for  $k = 1, 2, 3$ , be the linear functions satisfying  $L_k^0(0) = l_k$ ,  $L_k^0(1) = r_k$  for  $k = 1, 2$ ,  $L_3^0(0) = \phi_0$  and  $L_3^0(1) = 0$ . Lemma 3.1 guarantees the existence of functions  $L_k(X, Y, Z, \epsilon)$  for  $k = 1, 2, 3$  such that for each  $\epsilon > 0$  and  $Y^2 + Z^2 < g^2(X, \epsilon)$ ,  $L_k(X, 0, 0, \epsilon) = L_k^0$ ,  $L_k(0, Y, Z, \epsilon) =$

$L_k^0(0)$ ,  $L_k(1, Y, Z, \epsilon) = L_k^0(1)$ , and  $\frac{\partial}{\partial \mathbf{n}} L_k(X, Y, Z, \epsilon) = 0$  when  $(X, Y, Z) \in \hat{M}_\epsilon$ . For each  $\epsilon > 0$  and  $k = 1, 2, 3$ , introduce the functions  $L_k^\epsilon$  in terms of variables  $x, y$  and  $z$ :

$$L_k^\epsilon(x, y, z) = L_k(X, Y, Z, \epsilon) = L_k(x, g(x, \epsilon)y, g(x, \epsilon)z, \epsilon). \quad (3.2)$$

Set

$$\begin{aligned} u(x, y, z) &= L_1^\epsilon(x, y, z) - c_1(x, y, z), \\ v(x, y, z) &= L_2^\epsilon(x, y, z) - c_2(x, y, z), \\ \phi(x, y, z) &= L_3^\epsilon(x, y, z) - \Phi(x, y, z). \end{aligned}$$

Then, problem (2.4)-(2.5) is transformed into

$$\begin{aligned} \frac{1}{g^2} \nabla \cdot (g^2 J J^\tau \nabla (\phi - L_3^\epsilon)) &= -\lambda(\alpha_1(u - L_1^\epsilon) - \alpha_2(v - L_2^\epsilon)), \\ \frac{\partial u}{\partial t} &= \frac{D_1}{g^2} \nabla \cdot (g^2 J J^\tau \nabla (u - L_1^\epsilon) - \alpha_1(u - L_1^\epsilon(x)) g^2 J J^\tau \nabla (\phi - L_3^\epsilon)), \\ \frac{\partial v}{\partial t} &= \frac{D_2}{g^2} \nabla \cdot (g^2 J J^\tau \nabla (v - L_2^\epsilon) + \alpha_2(v - L_2^\epsilon(x)) g^2 J J^\tau \nabla (\phi - L_3^\epsilon)), \end{aligned} \quad (3.3)$$

with the homogeneous boundary conditions:

$$\begin{aligned} \phi|_{L \cup R} = u|_{L \cup R} = v|_{L \cup R} &= 0, \\ \langle \nabla \phi, J J^\tau \nu \rangle|_M = \langle \nabla u, J J^\tau \nu \rangle|_M = \langle \nabla v, J J^\tau \nu \rangle|_M &= 0. \end{aligned} \quad (3.4)$$

System (3.3) is supplemented with the initial conditions:

$$u(0) = u_0, \quad v(0) = v_0. \quad (3.5)$$

Introduce the subspace  $H_D^1(\Omega)$  of  $H^1(\Omega)$ :

$$H_D^1(\Omega) = \{u \in H^1(\Omega) : u|_{L \cup R} = 0\}.$$

Let  $M$  be the constant in Proposition 2.2 and let  $\Sigma_\epsilon$  be the subset of  $H_D^1(\Omega) \times H_D^1(\Omega)$  given by

$$\Sigma_\epsilon = \{(u, v) \in H_D^1(\Omega) \times H_D^1(\Omega) : \alpha_1 L_1^\epsilon - M \leq \alpha_1 u \leq \alpha_1 L_1^\epsilon, \alpha_2 L_2^\epsilon - M \leq \alpha_2 v \leq \alpha_2 L_2^\epsilon\}. \quad (3.6)$$

It follows from Proposition 2.2 that, if  $(u_0, v_0) \in \Sigma_\epsilon$ , then  $(u(t), v(t)) \in \Sigma_\epsilon$  for every  $t \geq 0$ . Throughout this paper, for every  $\epsilon > 0$ , we denote by  $S^\epsilon(t)_{t \geq 0}$  the solution operator associated with problem (3.3)-(3.5). We will use the same symbol  $\mathcal{A}_\epsilon$  to denote the global attractors of  $S^\epsilon(t)_{t \geq 0}$  and that of problem (2.4)-(2.5) when no confusion arises.

The corresponding one-dimensional limiting system (2.6) is transformed into

$$\begin{aligned} \frac{1}{g_0^2} \frac{\partial}{\partial x} \left( g_0^2 \frac{\partial}{\partial x} (\phi - L_3^0) \right) &= -\lambda (\alpha_1(u - L_1^0) - \alpha_2(v - L_2^0)), \\ \frac{\partial u}{\partial t} &= \frac{D_1}{g_0^2} \frac{\partial}{\partial x} \left( g_0^2 \frac{\partial}{\partial x} (u - L_1^0) - \alpha_1(u - L_1^0) g_0^2 \frac{\partial}{\partial x} (\phi - L_3^0) \right), \\ \frac{\partial v}{\partial t} &= \frac{D_2}{g_0^2} \frac{\partial}{\partial x} \left( g_0^2 \frac{\partial}{\partial x} (v - L_2^0) + \alpha_2(v - L_2^0) g_0^2 \frac{\partial}{\partial x} (\phi - L_3^0) \right), \end{aligned} \quad (3.7)$$

with the homogeneous Dirichlet boundary conditions

$$\phi = u = v = 0, \quad x = 0, 1, \quad (3.8)$$

and the initial conditions

$$u(0) = u_0, \quad \text{and } v(0) = v_0. \quad (3.9)$$

Since  $\tilde{\Sigma}_0$  is an invariant region for problem (2.6)-(2.7), we find that the one-dimensional problem (3.7)-(3.9) also has a positively invariant region which is given by

$$\Sigma_0 = \{(u, v) \in H_0^1(0, 1) \times H_0^1(0, 1) : \alpha_1 L_1^0 - M \leq \alpha_1 u \leq \alpha_1 L_1^0, \alpha_2 L_2^0 - M \leq \alpha_2 v \leq \alpha_2 L_2^0\}. \quad (3.10)$$

Similar to system (2.6)-(2.7), problem (3.7)-(3.9) is well-posed in  $\Sigma_0$ , that is, for each  $(u_0, v_0) \in \Sigma_0$ , there exists a unique solution  $(u, v)$  for problem (3.7)-(3.9) which is defined for all  $t \geq 0$  and  $(u, v) \in \mathcal{C}([0, \infty), \Sigma_0)$ . Further, the solutions are continuous in initial data with respect to the topology of  $H_0^1(0, 1) \times H_0^1(0, 1)$ . Therefore, there is a continuous dynamical system  $S^0(t)_{t \geq 0}$  associated with problem (3.7)-(3.9) such that for each  $t \geq 0$  and  $(u_0, v_0) \in \Sigma_0$ ,  $S^0(t)(u_0, v_0) = (u(t), v(t))$ , the solution of problem (3.7)-(3.9). When no confusion arises, we use the same symbol  $\mathcal{A}_0$  to denote the global attractors of  $S^0(t)_{t \geq 0}$  and problem (2.6)-(2.7).

### 3.2 Uniform estimates of global attractors

In this section, we derive uniform estimates of the global attractors  $\mathcal{A}_\epsilon$  in  $\epsilon$  which are necessary for establishing the upper semi-continuity of  $\mathcal{A}_\epsilon$  at  $\epsilon = 0$ . In what follows, we reformulate problem (3.3)-(3.5) as an abstract differential equation in  $H_D^1(\Omega) \times H_D^1(\Omega)$ .

Given  $\epsilon > 0$ , define an inner product  $(\cdot, \cdot)_{H_\epsilon}$  on  $L^2(\Omega)$  by

$$(v, w)_{H_\epsilon} = \int_{\Omega} \frac{g^2}{\epsilon^2} v w \, dx \, dy \, dz,$$

and a bilinear form  $a_\epsilon(\cdot, \cdot)$  on  $(H_D^1(\Omega))^2$  by

$$a_\epsilon(w_1, w_2) = (J^T \nabla w_1, J^T \nabla w_2)_{H_\epsilon} = \int_{\Omega} \frac{g^2}{\epsilon^2} J^T \nabla w_1 \cdot J^T \nabla w_2 \, dx \, dy \, dz.$$

In the sequel, we denote  $\|w\|_p$  the standard norm of  $w$  for  $w \in L^p(\Omega)$  or  $w \in L^p([0, 1])$ ,  $\|w\|_{H^s}$  the standard norm of  $w$  for  $w \in H^s(\Omega)$  or  $w \in H^s([0, 1])$ . Also, denote  $H_\epsilon$  the space  $L^2(\Omega)$  with the inner product  $(\cdot, \cdot)_{H_\epsilon}$ , and  $X_\epsilon$  the space  $H_D^1(\Omega)$  with the norm

$$\|w\|_{X_\epsilon} = \left( \|\nabla w\|_2^2 + \frac{1}{\epsilon^2} \|w_y\|_2^2 + \frac{1}{\epsilon^2} \|w_z\|_2^2 \right)^{1/2}.$$

Since Poincare inequality holds in  $H_D^1(\Omega)$ , the norm  $\|w\|_{X_\epsilon}$  for  $w \in H_D^1(\Omega)$  is equivalent to the norm given by

$$\left( \|w\|_{H^1}^2 + \frac{1}{\epsilon^2} \|w_y\|_2^2 + \frac{1}{\epsilon^2} \|w_z\|_2^2 \right)^{1/2}.$$

Due to assumption (2.1), there exist positive constants  $C_1, C_2, C_3$  (independent of  $\epsilon$ ) and  $\epsilon_1$  such that, for all  $0 < \epsilon \leq \epsilon_1$  and  $x \in (0, 1)$ ,

$$\frac{|g_x|}{g} \leq C_1, \quad C_2 \leq \frac{g}{\epsilon} \leq C_3. \quad (3.11)$$

Consequently,  $\sqrt{a_\epsilon(w, w)}$  is equivalent to the norm  $\|w\|_{X_\epsilon}$ , that is,

$$C_4 \|w\|_{X_\epsilon}^2 \leq a_\epsilon(w, w) \leq C_5 \|w\|_{X_\epsilon}^2 \quad (3.12)$$

for some constants  $C_4$  and  $C_5$  (independent of  $\epsilon$ ). It follows from (3.12) that for each  $\epsilon > 0$ , the triple  $\{H_D^1(\Omega), H_\epsilon, a_\epsilon(\cdot, \cdot)\}$  defines a unique unbounded operator  $\mathcal{L}_\epsilon$  on  $H_D^1(\Omega)$  with domain  $D(\mathcal{L}_\epsilon)$  in the following way: an element  $u \in H_D^1(\Omega)$  belongs to  $D(\mathcal{L}_\epsilon)$  if  $a_\epsilon(u, v)$  is continuous in  $v \in H_D^1(\Omega)$  for the topology induced from  $H_\epsilon$  and  $(\mathcal{L}_\epsilon u, v)_{H_\epsilon} = a_\epsilon(u, v)$  for  $(u, v) \in D(\mathcal{L}_\epsilon) \times H_D^1(\Omega)$ . In fact,

$$D(\mathcal{L}_\epsilon) = \{u \in H_D^1(\Omega) : \mathcal{L}_\epsilon u \in H_\epsilon\},$$

and for every  $u \in D(\mathcal{L}_\epsilon)$ ,

$$\mathcal{L}_\epsilon u = -\frac{1}{g^2} \nabla \cdot (g^2 J J^T \nabla u).$$

Since the operator  $\mathcal{L}_\epsilon$  is self-adjoint on  $H_\epsilon$  and positive, the fractional power  $\mathcal{L}_\epsilon^{1/2}$  is well-defined with domain  $D(\mathcal{L}_\epsilon^{1/2}) = H_D^1(\Omega)$ , and for  $u \in H_D^1(\Omega)$ ,

$$\|\mathcal{L}_\epsilon^{1/2} u\|_{H_\epsilon}^2 = a_\epsilon(u, u).$$

In view of (3.12) there exist  $C_6$  and  $C_7$  such that

$$C_6 \|u\|_{X_\epsilon} \leq \|\mathcal{L}_\epsilon^{1/2} u\|_{H_\epsilon} \leq C_7 \|u\|_{X_\epsilon}. \quad (3.13)$$

With the above notations, system (3.3) can be rewritten as

$$\begin{aligned}\mathcal{L}_\epsilon \phi &= \lambda \alpha_1 (u - L_1^\epsilon) - \lambda \alpha_2 (v - L_2^\epsilon) - \frac{1}{g^2} \nabla \cdot (g^2 J J^\tau \nabla L_3^\epsilon), \\ \frac{\partial u}{\partial t} + D_1 \mathcal{L}_\epsilon u &= -\frac{D_1}{g^2} \nabla \cdot (g^2 J J^\tau \nabla L_1^\epsilon + \alpha_1 (u - L_1^\epsilon) g^2 J J^\tau \nabla (\phi - L_3^\epsilon)), \\ \frac{\partial v}{\partial t} + D_2 \mathcal{L}_\epsilon v &= -\frac{D_2}{g^2} \nabla \cdot (g^2 J J^\tau \nabla L_2^\epsilon - \alpha_2 (v - L_2^\epsilon) g^2 J J^\tau \nabla (\phi - L_3^\epsilon)).\end{aligned}\tag{3.14}$$

By the construction of functions  $L_k^\epsilon$  ( $k = 1, 2, 3$ ), there exists  $\epsilon_1 > 0$  such that for any  $0 < \epsilon \leq \epsilon_1$ , the following uniform bounds in  $\epsilon$  hold:

$$\|L_k^\epsilon\|_\infty + \|L_k^\epsilon\|_{H_\epsilon} + \|J^\tau \nabla L_k^\epsilon\|_{H_\epsilon} + \|J J^\tau \nabla L_k^\epsilon\|_{H_\epsilon} + \left\| \frac{1}{g^2} \nabla \cdot (g^2 J J^\tau \nabla L_k^\epsilon) \right\|_{H_\epsilon} \leq C, \tag{3.15}$$

where  $C$  is independent of  $\epsilon$ . Then it follows from the positive invariance of  $\Sigma_\epsilon$  that there exists a constant  $C$  (independent of  $\epsilon$ ) such that for any initial datum  $(u_0, v_0) \in \Sigma_\epsilon$ , the solution  $(u, v)$  of problem (3.3)-(3.5) satisfies, for all  $t \geq 0$ :

$$\|u(t)\|_\infty + \|v(t)\|_\infty \leq C \quad \text{and} \quad \|u(t)\|_{H_\epsilon} + \|v(t)\|_{H_\epsilon} \leq C. \tag{3.16}$$

Next, we start to derive uniform estimates of solutions in  $\epsilon$  in the space  $H_D^1(\Omega) \times H_D^1(\Omega)$ .

**Lemma 3.2.** *There exist a constant  $C$  (independent of  $\epsilon$ ) and  $\epsilon_1 > 0$  such that for any  $0 < \epsilon \leq \epsilon_1$  and  $(u_0, v_0) \in \Sigma_\epsilon$ , the solution  $(u, v)$  of problem (3.3)-(3.5) satisfies, for all  $t \geq 0$ :*

$$\int_t^{t+1} (\|u(t)\|_{X_\epsilon} + \|v(t)\|_{X_\epsilon}) dt \leq C.$$

*Proof.* Taking the inner product of the first equation in (3.14) with  $\phi$  in  $H_\epsilon$ , we find that

$$\|\mathcal{L}_\epsilon^{\frac{1}{2}} \phi\|_{H_\epsilon}^2 = \lambda \alpha_1 (u - L_1^\epsilon, \phi)_{H_\epsilon} - \lambda \alpha_2 (v - L_2^\epsilon, \phi)_{H_\epsilon} + (J^\tau \nabla L_3^\epsilon, J^\tau \nabla \phi)_{H_\epsilon}.$$

By (3.15) and (3.16) we have

$$\begin{aligned}\|\mathcal{L}_\epsilon^{\frac{1}{2}} \phi\|_{H_\epsilon}^2 &\leq \lambda \alpha_1 (\|u\|_{H_\epsilon} + \|L_1^\epsilon\|_{H_\epsilon}) \|\phi\|_{H_\epsilon} + \lambda \alpha_2 (\|v\|_{H_\epsilon} + \|L_2^\epsilon\|_{H_\epsilon}) \|\phi\|_{H_\epsilon} + \|J^\tau \nabla L_3^\epsilon\|_{H_\epsilon} \|\mathcal{L}_\epsilon^{\frac{1}{2}} \phi\|_{H_\epsilon} \\ &\leq C \|\mathcal{L}_\epsilon^{\frac{1}{2}} \phi\|_{H_\epsilon} \leq \frac{1}{2} \|\mathcal{L}_\epsilon^{\frac{1}{2}} \phi\|_{H_\epsilon}^2 + \frac{1}{2} C^2,\end{aligned}$$

which implies that

$$\|\mathcal{L}_\epsilon^{\frac{1}{2}} \phi\|_{H_\epsilon} \leq C. \tag{3.17}$$

Now, taking the inner product of the second equation in (3.14) with  $u$  in  $H_\epsilon$ , we get

$$\frac{1}{2} \frac{d}{dt} \|u\|_{H_\epsilon}^2 + D_1 \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon}^2 = D_1 (J^\tau \nabla L_1^\epsilon, J^\tau \nabla u)_{H_\epsilon} + D_1 \alpha_1 ((u - L_1^\epsilon) J^\tau \nabla (\phi - L_3^\epsilon), J^\tau \nabla u)_{H_\epsilon}.$$

It follows from (3.15)-(3.17) that the right-hand side of the above is bounded by

$$\begin{aligned} C_1 \|J^\tau \nabla L_1^\epsilon\|_{H_\epsilon} \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon} + D_1 \alpha_1 (\|u\|_\infty + \|L_1^\epsilon\|_\infty) (\|\mathcal{L}_\epsilon^{\frac{1}{2}} \phi\|_{H_\epsilon} + \|J^\tau \nabla L_3^\epsilon\|_{H_\epsilon}) \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon} \\ \leq C \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon} \leq \frac{1}{2} D_1 \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon}^2 + C_1. \end{aligned}$$

Therefore,

$$\frac{d}{dt} \|u\|_{H_\epsilon}^2 + D_1 \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon}^2 \leq C_2. \quad (3.18)$$

Similarly,

$$\frac{d}{dt} \|v\|_{H_\epsilon}^2 + D_2 \|\mathcal{L}_\epsilon^{\frac{1}{2}} v\|_{H_\epsilon}^2 \leq C_3. \quad (3.19)$$

Hence, for all  $t \geq 0$ :

$$\frac{d}{dt} (\|u\|_{H_\epsilon}^2 + \|v\|_{H_\epsilon}^2) + C_4 \left( \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon}^2 + \|\mathcal{L}_\epsilon^{\frac{1}{2}} v\|_{H_\epsilon}^2 \right) \leq C_2 + C_3,$$

which, along (3.13) and (3.16), implies Lemma 3.2.  $\square$

**Lemma 3.3.** *There exist positive constants  $\epsilon_1$  and  $C$  such that for any  $0 < \epsilon \leq \epsilon_1$  and  $(u_0, v_0) \in \Sigma_\epsilon$ , the solution  $(u, v)$  of problem (3.3)-(3.5) satisfies, for all  $t \geq 1$ :*

$$\|\mathcal{L}_\epsilon \phi(t)\|_{X_\epsilon} + \|u(t)\|_{X_\epsilon} + \|v(t)\|_{X_\epsilon} \leq C.$$

*Proof.* By (3.15), (3.16) and the first equation in (3.14) we get

$$\|\mathcal{L}_\epsilon \phi\|_{H_\epsilon} \leq C \left( \|u\|_{H_\epsilon} + \|v\|_{H_\epsilon} + \|L_1^\epsilon\|_{H_\epsilon} + \|L_2^\epsilon\|_{H_\epsilon} + \left\| \frac{1}{g^2} \nabla \cdot (g^2 J J^\tau \nabla L_3^\epsilon) \right\|_{H_\epsilon} \right) \leq C. \quad (3.20)$$

Taking the inner product of the second equation in (3.14) with  $\mathcal{L}_\epsilon u$  in  $H_\epsilon$ , we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon}^2 + D_1 \|\mathcal{L}_\epsilon u\|_{H_\epsilon}^2 = - \left( \frac{D_1}{g^2} \nabla \cdot (g^2 J J^\tau \nabla L_1^\epsilon), \mathcal{L}_\epsilon u \right)_{H_\epsilon} \\ - \left( \frac{D_1 \alpha_1}{g^2} \nabla \cdot ((u - L_1^\epsilon) g^2 J J^\tau \nabla (\phi - L_3^\epsilon)), \mathcal{L}_\epsilon u \right)_{H_\epsilon}. \end{aligned} \quad (3.21)$$

By (3.15), the first term on the right-hand side of (3.21) is bounded by

$$\left| \left( \frac{D_1}{g^2} \nabla \cdot (g^2 J J^\tau \nabla L_1^\epsilon), \mathcal{L}_\epsilon u \right)_{H_\epsilon} \right| \leq D_1 \left\| \frac{1}{g^2} \nabla \cdot (g^2 J J^\tau \nabla L_1^\epsilon) \right\|_{H_\epsilon} \|\mathcal{L}_\epsilon u\|_{H_\epsilon} \leq \frac{1}{4} D_1 \|\mathcal{L}_\epsilon u\|_{H_\epsilon}^2 + C. \quad (3.22)$$

For the second term on the right-hand side of (3.21), we have

$$\begin{aligned} - \left( \frac{D_1 \alpha_1}{g^2} \nabla \cdot ((u - L_1^\epsilon) g^2 J J^\tau \nabla (\phi - L_3^\epsilon)), \mathcal{L}_\epsilon u \right)_{H_\epsilon} \\ = -D_1 \alpha_1 (\nabla(u - L_1^\epsilon) \cdot J J^\tau \nabla(\phi - L_3^\epsilon), \mathcal{L}_\epsilon u)_{H_\epsilon} \\ - D_1 \alpha_1 \left( (u - L_1^\epsilon) \frac{1}{g^2} \nabla \cdot (g^2 J J^\tau \nabla(\phi - L_3^\epsilon)), \mathcal{L}_\epsilon u \right)_{H_\epsilon}. \end{aligned} \quad (3.23)$$

Using (3.15) and (3.20), the first term on the right-hand side of (3.23) is bounded by

$$\begin{aligned}
& D_1 \alpha_1 |(\nabla(u - L_1^\epsilon) \cdot J J^T \nabla(\phi - L_3^\epsilon), \mathcal{L}_\epsilon u)_{H_\epsilon}| \\
& \leq D_1 \alpha_1 \|\nabla(u - L_1^\epsilon)\|_3 \left\| \frac{g^2}{\epsilon^2} J J^T \nabla(\phi - L_3^\epsilon) \right\|_6 \|\mathcal{L}_\epsilon u\|_2 \\
& \leq C \|\nabla(u - L_1^\epsilon)\|_2^{\frac{1}{2}} \|\nabla(u - L_1^\epsilon)\|_{H^1}^{\frac{1}{2}} \left\| \frac{g^2}{\epsilon^2} J J^T \nabla(\phi - L_3^\epsilon) \right\|_{H^1} \|\mathcal{L}_\epsilon u\|_2 \\
& \leq \left( \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon} + \|J^T \nabla L_1^\epsilon\|_{H_\epsilon} \right)^{\frac{1}{2}} \left( \|\mathcal{L}_\epsilon u\|_{H_\epsilon} + \|J^T \nabla L_1^\epsilon\|_{H_\epsilon} + \left\| \frac{1}{g^2} \nabla \cdot (g^2 J J^T \nabla L_1^\epsilon) \right\|_{H_\epsilon} \right)^{\frac{1}{2}} \\
& \times \left( \|\mathcal{L}_\epsilon \phi\|_{H_\epsilon} + \|J J^T \nabla L_3^\epsilon\|_{H_\epsilon} + \left\| \frac{1}{g^2} \nabla \cdot (g^2 J J^T \nabla L_3^\epsilon) \right\|_{H_\epsilon} \right) \|\mathcal{L}_\epsilon u\|_{H_\epsilon} \\
& \leq C \left( \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon} + C \right)^{\frac{1}{2}} (\|\mathcal{L}_\epsilon u\|_{H_\epsilon} + C)^{\frac{1}{2}} \|\mathcal{L}_\epsilon u\|_{H_\epsilon} \\
& \leq \frac{1}{8} D_1 \|\mathcal{L}_\epsilon u\|_{H_\epsilon}^2 + C \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon}^2 + C.
\end{aligned} \tag{3.24}$$

The second term on the right-hand side of (3.23) can be estimated as

$$\begin{aligned}
& D_1 \alpha_1 \left| \left( (u - L_1^\epsilon) \frac{1}{g^2} \nabla \cdot (g^2 J J^T \nabla(\phi - L_3^\epsilon)), \mathcal{L}_\epsilon u \right)_{H_\epsilon} \right| \\
& \leq D_1 \alpha_1 (\|u\|_\infty + \|L_1^\epsilon\|_\infty) \left\| \frac{1}{g^2} \nabla \cdot (g^2 J J^T \nabla(\phi - L_3^\epsilon)) \right\|_{H_\epsilon} \|\mathcal{L}_\epsilon u\|_{H_\epsilon} \\
& \leq D_1 \alpha_1 (\|u\|_\infty + \|L_1^\epsilon\|_\infty) \left( \|\mathcal{L}_\epsilon \phi\|_{H_\epsilon} + \left\| \frac{1}{g^2} \nabla \cdot (g^2 J J^T \nabla L_3^\epsilon) \right\|_{H_\epsilon} \right) \|\mathcal{L}_\epsilon u\|_{H_\epsilon} \\
& \leq C \|\mathcal{L}_\epsilon u\|_{H_\epsilon} \leq \frac{1}{8} D_1 \|\mathcal{L}_\epsilon u\|_{H_\epsilon}^2 + C.
\end{aligned} \tag{3.25}$$

Combining the estimates (3.23)-(3.25), we obtain

$$\left| \left( \frac{D_1 \alpha_1}{g^2} \nabla \cdot ((u - L_1^\epsilon) g^2 J J^T \nabla(\phi - L_3^\epsilon)), \mathcal{L}_\epsilon u \right)_{H_\epsilon} \right| \leq \frac{1}{4} D_1 \|\mathcal{L}_\epsilon u\|_{H_\epsilon}^2 + C \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon}^2 + C. \tag{3.26}$$

It follows from (3.21), (3.22) and (3.26) that, for all  $t \geq 0$ ,

$$\frac{d}{dt} \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon}^2 + D_1 \|\mathcal{L}_\epsilon u\|_{H_\epsilon}^2 \leq C_1 \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon}^2 + C_2. \tag{3.27}$$

Similarly, for all  $t \geq 0$ ,

$$\frac{d}{dt} \|\mathcal{L}_\epsilon^{\frac{1}{2}} v\|_{H_\epsilon}^2 + D_2 \|\mathcal{L}_\epsilon v\|_{H_\epsilon}^2 \leq C_1 \|\mathcal{L}_\epsilon^{\frac{1}{2}} v\|_{H_\epsilon}^2 + C_2. \tag{3.28}$$

Hence, we have, for all  $t \geq 0$ ,

$$\frac{d}{dt} \left( \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon}^2 + \|\mathcal{L}_\epsilon^{\frac{1}{2}} v\|_{H_\epsilon}^2 \right) + C_3 (\|\mathcal{L}_\epsilon u\|_{H_\epsilon}^2 + \|\mathcal{L}_\epsilon v\|_{H_\epsilon}^2) \leq C_1 \left( \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon}^2 + \|\mathcal{L}_\epsilon^{\frac{1}{2}} v\|_{H_\epsilon}^2 \right) + C_2, \tag{3.29}$$



which, along with Lemma 3.2 and the uniform Gronwall's lemma, implies that, for all  $t \geq 1$ ,

$$\|\mathcal{L}_\epsilon^{\frac{1}{2}}u(t)\|_{H_\epsilon}^2 + \|\mathcal{L}_\epsilon^{\frac{1}{2}}v(t)\|_{H_\epsilon}^2 \leq C.$$

The above estimate and the first equation in (3.14) conclude the proof.  $\square$

Applying Gronwall's lemma to (3.29) for  $t \in (0, 1)$ , then by Lemma 3.3 and the first equation in (3.14) we find that there exists  $\epsilon_1 > 0$  such that, for any  $R > 0$ , there exists  $K$  depending on  $R$  such that for any  $0 < \epsilon \leq \epsilon_1$  and  $(u_0, v_0) \in \Sigma_\epsilon$  with  $\|(u_0, v_0)\|_{X_\epsilon \times X_\epsilon} \leq R$ , the following holds:

$$\|\mathcal{L}_\epsilon \phi(t)\|_{X_\epsilon} + \|u(t)\|_{X_\epsilon} + \|v(t)\|_{X_\epsilon} \leq K, \quad \text{for } t \geq 0. \quad (3.30)$$

An immediate consequence of Lemma 3.3 also shows that all the global attractors  $\mathcal{A}_\epsilon$  are uniformly bounded in  $\epsilon$  in the space  $H_D^1(\Omega) \times H_D^1(\Omega)$ , that is, the following statement is true.

**Proposition 3.4.** *There exist positive constants  $\epsilon_1$  and  $C$  such that for all  $0 < \epsilon \leq \epsilon_1$  and  $(u, v) \in \mathcal{A}_\epsilon$ , the following holds:*

$$\|(u, v)\|_{X_\epsilon \times X_\epsilon} \leq C.$$

The following is an analogue of Lemma 3.3 for the limiting system (3.7)-(3.9).

**Lemma 3.5.** *There exists  $C > 0$  such that for any  $(u_0, v_0) \in \Sigma_0$ , the solution  $(u, v)$  of problem (3.7)-(3.9) satisfies, for all  $t \geq 1$ :*

$$\|u(t)\|_{H^1} + \|v(t)\|_{H^1} \leq C.$$

*In addition, there exists  $K$  depending on  $R$  when  $\|(u_0, v_0)\|_{H^1 \times H^1} \leq R$  such that for all  $t \geq 0$ :*

$$\|u(t)\|_{H^1} + \|v(t)\|_{H^1} \leq K.$$

Next, we establish estimates on time derivatives of solutions for both the three-dimensional system and the one-dimensional limiting system.

**Lemma 3.6.** *There exists  $\epsilon_1 > 0$  such that for any  $R > 0$ , there exists  $K$  depending only on  $R$  such that for any  $0 < \epsilon \leq \epsilon_1$  and  $(u_0, v_0) \in \Sigma_\epsilon$  with  $\|(u_0, v_0)\|_{X_\epsilon \times X_\epsilon} \leq R$ , the solution  $(u, v)$  of problem (3.3)-(3.5) satisfies*

$$t^2 \left( \|\mathcal{L}_\epsilon \frac{\partial \phi}{\partial t}\|_{H_\epsilon}^2 + \|\frac{\partial u}{\partial t}\|_{H_\epsilon}^2 + \|\frac{\partial v}{\partial t}\|_{H_\epsilon}^2 \right) + \int_0^t s^2 \left( \|\frac{\partial u}{\partial s}\|_{X_\epsilon}^2 + \|\frac{\partial v}{\partial s}\|_{X_\epsilon}^2 \right) ds \leq K e^{Kt}, \quad t \geq 0.$$

*Proof.* Denote by

$$\tilde{\phi} = \frac{\partial \phi}{\partial t}, \quad \tilde{u} = \frac{\partial u}{\partial t}, \quad \tilde{v} = \frac{\partial v}{\partial t}.$$

Differentiating (3.14) with respect to  $t$ , we get

$$\begin{aligned}\mathcal{L}_\epsilon \tilde{\phi} &= \lambda \alpha_1 \tilde{u} - \lambda \alpha_2 \tilde{v}, \\ \frac{\partial \tilde{u}}{\partial t} + D_1 \mathcal{L}_\epsilon \tilde{u} &= -\frac{D_1}{g^2} \nabla \cdot \left( \alpha_1 \tilde{u} g^2 J J^\tau \nabla (\phi - L_3^\epsilon) + \alpha_1 (u - L_1^\epsilon) g^2 J J^\tau \nabla \tilde{\phi} \right), \\ \frac{\partial \tilde{v}}{\partial t} + D_2 \mathcal{L}_\epsilon \tilde{v} &= -\frac{D_2}{g^2} \nabla \cdot \left( \alpha_2 \tilde{v} g^2 J J^\tau \nabla (\phi - L_3^\epsilon) + \alpha_2 (v - L_2^\epsilon) g^2 J J^\tau \nabla \tilde{\phi} \right).\end{aligned}$$

From the above system, one derives

$$\begin{aligned}\mathcal{L}_\epsilon(t\tilde{\phi}) &= \lambda \alpha_1 t\tilde{u} - \lambda \alpha_2 t\tilde{v}, \\ \frac{\partial}{\partial t}(t\tilde{u}) + D_1 \mathcal{L}_\epsilon(t\tilde{u}) &= \tilde{u} - \frac{D_1}{g^2} \nabla \cdot \left( \alpha_1(t\tilde{u}) g^2 J J^\tau \nabla (\phi - L_3^\epsilon) + \alpha_1 (u - L_1^\epsilon) g^2 J J^\tau \nabla(t\tilde{\phi}) \right), \\ \frac{\partial}{\partial t}(t\tilde{v}) + D_2 \mathcal{L}_\epsilon(t\tilde{v}) &= \tilde{v} + \frac{D_2}{g^2} \nabla \cdot \left( \alpha_2(t\tilde{v}) g^2 J J^\tau \nabla (\phi - L_3^\epsilon) + \alpha_2 (v - L_2^\epsilon) g^2 J J^\tau \nabla(t\tilde{\phi}) \right).\end{aligned}\tag{3.31}$$

The first equation in (3.31) gives

$$\|\mathcal{L}_\epsilon(t\tilde{\phi})\|_{H_\epsilon} \leq C(\|t\tilde{u}\|_{H_\epsilon} + \|t\tilde{v}\|_{H_\epsilon}).\tag{3.32}$$

Taking the inner product of the second equation in (3.31) with  $t\tilde{u}$  in  $H_\epsilon$ , we have

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \|t\tilde{u}\|_{H_\epsilon}^2 + D_1 \|\mathcal{L}_\epsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\epsilon}^2 &= D_1 \alpha_1 \int \frac{g^2}{\epsilon^2} t\tilde{u} J^\tau \nabla (\phi - L_3^\epsilon) \cdot J^\tau \nabla(t\tilde{u}) \\ &\quad + D_1 \alpha_1 \int \frac{g^2}{\epsilon^2} (u - L_1^\epsilon) J^\tau \nabla(t\tilde{\phi}) \cdot J^\tau \nabla(t\tilde{u}) + t \|\tilde{u}\|_{H_\epsilon}^2.\end{aligned}\tag{3.33}$$

By (3.30), the first term on the right-hand side of (3.33) is bounded by

$$\begin{aligned}C \|t\tilde{u}\|_3 \|J^\tau \nabla(\phi - L_3^\epsilon)\|_6 \|J^\tau \nabla(t\tilde{u})\|_2 &\leq C \|t\tilde{u}\|_{\frac{1}{2}}^{\frac{1}{2}} \|t\tilde{u}\|_{H^1}^{\frac{1}{2}} \|J^\tau \nabla(\phi - L_3^\epsilon)\|_{H^1} \|J^\tau \nabla(t\tilde{u})\|_2 \\ &\leq C \|t\tilde{u}\|_{H_\epsilon}^{\frac{1}{2}} \|\mathcal{L}_\epsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\epsilon}^{\frac{3}{2}} (\|\mathcal{L}_\epsilon \phi\|_{H_\epsilon} + \|\mathcal{L}_\epsilon L_3^\epsilon\|_{H_\epsilon}) \\ &\leq C \|t\tilde{u}\|_{H_\epsilon}^{\frac{1}{2}} \|\mathcal{L}_\epsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\epsilon}^{\frac{3}{2}} \leq \frac{1}{8} D_1 \|\mathcal{L}_\epsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\epsilon}^2 + C \|t\tilde{u}\|_{H_\epsilon}^2.\end{aligned}\tag{3.34}$$

By (3.32), the second term on the right-hand side of (3.33) is less than

$$\begin{aligned}C \|u - L_1^\epsilon\|_\infty \|J^\tau \nabla(t\tilde{\phi})\|_2 \|J^\tau \nabla(t\tilde{u})\|_2 &\leq \frac{1}{8} D_1 \|\mathcal{L}_\epsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\epsilon}^2 + C \|\mathcal{L}_\epsilon^{\frac{1}{2}}(t\tilde{\phi})\|_{H_\epsilon}^2 \\ &\leq \frac{1}{8} D_1 \|\mathcal{L}_\epsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\epsilon}^2 + C (\|t\tilde{u}\|_{H_\epsilon}^2 + \|t\tilde{v}\|_{H_\epsilon}^2).\end{aligned}\tag{3.35}$$

Multiplying the second equation in (3.14) by  $t\tilde{u}$ , after simple computations, we find that the last term on the right-hand side of (3.33) satisfies

$$\begin{aligned}t \|\tilde{u}\|_{H_\epsilon}^2 &\leq C \|\mathcal{L}_\epsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\epsilon} \left( \|\mathcal{L}_\epsilon^{\frac{1}{2}} u\|_{H_\epsilon} + \|\mathcal{L}_\epsilon^{\frac{1}{2}} L_1^\epsilon\|_{H_\epsilon} + \|\mathcal{L}_\epsilon^{\frac{1}{2}} \phi\|_{H_\epsilon} + \|\mathcal{L}_\epsilon^{\frac{1}{2}} L_3^\epsilon\|_{H_\epsilon} \right) \\ &\leq \frac{1}{8} D_1 \|\mathcal{L}_\epsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\epsilon}^2 + C.\end{aligned}\tag{3.36}$$

Combining the estimates in (3.33)-(3.36), we get

$$\frac{d}{dt} \|t\tilde{u}\|_{H_\epsilon}^2 + D_1 \|\mathcal{L}_\epsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\epsilon}^2 \leq C (\|t\tilde{u}\|_{H_\epsilon}^2 + \|t\tilde{v}\|_{H_\epsilon}^2) + C. \quad (3.37)$$

Similarly,

$$\frac{d}{dt} \|t\tilde{v}\|_{H_\epsilon}^2 + D_1 \|\mathcal{L}_\epsilon^{\frac{1}{2}}(t\tilde{v})\|_{H_\epsilon}^2 \leq C (\|t\tilde{u}\|_{H_\epsilon}^2 + \|t\tilde{v}\|_{H_\epsilon}^2) + C. \quad (3.38)$$

Finally, from (3.37)-(3.38), we have

$$\frac{d}{dt} (\|t\tilde{u}\|_{H_\epsilon}^2 + \|t\tilde{v}\|_{H_\epsilon}^2) + C_1 \left( \|\mathcal{L}_\epsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\epsilon}^2 + \|\mathcal{L}_\epsilon^{\frac{1}{2}}(t\tilde{v})\|_{H_\epsilon}^2 \right) \leq C (\|t\tilde{u}\|_{H_\epsilon}^2 + \|t\tilde{v}\|_{H_\epsilon}^2) + C,$$

which, along with Gronwall's lemma, concludes the proof.  $\square$

We now describe the analogue of Lemma 3.6 for the one-dimensional limiting system (3.7)-(3.9).

**Lemma 3.7.** *Given  $R > 0$ , there exists  $K$  depending only on  $R$  such that for any  $(u_0, v_0) \in \Sigma_0$  with  $\|(u_0, v_0)\|_{H^1 \times H^1} \leq R$ , the solution  $(u, v)$  of problem (3.7)-(3.9) satisfies*

$$t^2 \left( \left\| \frac{\partial \phi}{\partial t} \right\|_{H^2}^2 + \left\| \frac{\partial u}{\partial t} \right\|_2^2 + \left\| \frac{\partial v}{\partial t} \right\|_2^2 \right) + \int_0^t s^2 \left( \left\| \frac{\partial u}{\partial s} \right\|_{H^1}^2 + \left\| \frac{\partial v}{\partial s} \right\|_{H^1}^2 \right) ds \leq K e^{Kt}, \quad t \geq 0.$$

*Proof.* The proof is similar to that of Lemma 3.6 but simpler, and therefore omitted here.  $\square$

### 3.3 Upper Semicontinuity

In this section, we establish the upper semicontinuity of global attractors  $\mathcal{A}_\epsilon$  at  $\epsilon = 0$ . We first compare the solutions of the three-dimensional problem (3.3)-(3.5) and the one-dimensional limiting problem (3.7)-(3.9), and then establish the relationships between the global attractors of the two dynamical systems.

In what follows, we reformulate limiting system (3.7) as an operator equation. Let  $H_0$  be the  $L^2(0, 1)$  space with the inner product  $(\cdot, \cdot)_{H_0}$  given by

$$(u, v)_{H_0} = \int_0^1 g_0^2 uv \, dx,$$

and let  $a_0(\cdot, \cdot)$  be the bilinear form on  $(H_0^1(0, 1))^2$ :

$$a_0(w_1, w_2) = \left( \frac{dw_1}{dx}, \frac{dw_2}{dx} \right)_{H_0} = \int_0^1 g_0^2 \frac{dw_1}{dx} \frac{dw_2}{dx} \, dx.$$

For  $f \in L^2(\Omega)$ , let  $M(f) \in L^2(0, 1)$  be the function:

$$(M(f))(x) = \frac{1}{\pi} \int_{\mathbb{D}} f(x, y, z) \, dy \, dz.$$

**Lemma 3.8.** *Suppose  $f \in H^1(\Omega)$ . Then we have*

$$\|f - M(f)\|_{H_\epsilon} \leq C\epsilon \|f\|_{X_\epsilon}.$$

*Proof.* Notice that

$$\|f - M(f)\|_2^2 = \int_{x=0}^1 \int_{\mathbb{D}} \left| f(x, y, z) - \frac{1}{\pi} \int_{\mathbb{D}} f(x, u, v) du dv \right|^2 dy dz dx. \quad (3.39)$$

Using the identity

$$\begin{aligned} f(x, r \cos \theta, r \sin \theta) &= f(x, \rho \cos \phi, \rho \sin \phi) - \int_{\theta}^{\phi} \frac{\partial}{\partial t} f(x, \rho \cos t, \rho \sin t) dt \\ &\quad - \int_r^{\rho} \frac{\partial}{\partial \tau} f(x, \tau \cos \theta, \tau \sin \theta) d\tau, \end{aligned}$$

one can write

$$\begin{aligned} \frac{1}{\pi} \int_{r=0}^1 \int_{\theta=0}^{2\pi} f(x, r \cos \theta, r \sin \theta) r dr d\theta &= f(x, \rho \cos \phi, \rho \sin \phi) \\ &\quad - \frac{1}{\pi} \int_{r=0}^1 \int_{\theta=0}^{2\pi} \left( \int_{t=\theta}^{\phi} \left( -\rho \sin t \frac{\partial f}{\partial y}(x, \rho \cos t, \rho \sin t) + \rho \cos t \frac{\partial f}{\partial z}(x, \rho \cos t, \rho \sin t) \right) dt \right) r dr d\theta \\ &\quad - \frac{1}{\pi} \int_{r=0}^1 \int_{\theta=0}^{2\pi} \left( \int_{\tau=r}^{\rho} \left( \cos \theta \frac{\partial f}{\partial y}(x, \tau \cos \theta, \tau \sin \theta) + \sin \theta \frac{\partial f}{\partial z}(x, \tau \cos \theta, \tau \sin \theta) \right) d\tau \right) r dr d\theta \end{aligned}$$

Then, after simple computations, Lemma 3.8 follows from (3.39) and the above.  $\square$

Let  $(\psi, P, Q) \in (H_D^1(\Omega))^3$  and let  $(\phi_\epsilon, u_\epsilon, v_\epsilon)$  be a solution of system (3.3). In view of the boundary condition (3.4) and the choices of  $L_k^\epsilon$  for  $k = 1, 2, 3$ , we have

$$\begin{aligned} -a_\epsilon(\phi_\epsilon - L_3^\epsilon, \psi) &= -\lambda\alpha_1(u_\epsilon - L_1^\epsilon, \psi)_{H_\epsilon} + \lambda\alpha_2(v_\epsilon - L_2^\epsilon, \psi)_{H_\epsilon}, \\ \frac{1}{D_1} \left( \frac{\partial u_\epsilon}{\partial t}, P \right)_{H_\epsilon} &= -a_\epsilon(u_\epsilon - L_1^\epsilon, P) + \alpha_1 \left( (u_\epsilon - L_1^\epsilon) \mathcal{L}_\epsilon^{1/2}(\phi_\epsilon - L_3^\epsilon), \mathcal{L}_\epsilon^{1/2} P \right)_{H_\epsilon}, \\ \frac{1}{D_2} \left( \frac{\partial v_\epsilon}{\partial t}, Q \right)_{H_\epsilon} &= -a_\epsilon(v_\epsilon - L_2^\epsilon, Q) - \alpha_2 \left( (v_\epsilon - L_2^\epsilon) \mathcal{L}_\epsilon^{1/2}(\phi_\epsilon - L_3^\epsilon), \mathcal{L}_\epsilon^{1/2} Q \right)_{H_\epsilon}. \end{aligned} \quad (3.40)$$

Let  $(\phi, u, v)$  be the solution of the limiting system (3.7). View  $(\phi, u, v)$  as an element in  $(H_D^1(\Omega))^3$ . Then a direct computation yields that, for  $(\psi, P, Q) \in (H_D^1(\Omega))^3$ ,

$$\begin{aligned} -a_\epsilon(\phi - L_3^0, \psi) &= -\lambda\alpha_1(u - L_1^0, \psi)_{H_\epsilon} + \lambda\alpha_2(v - L_2^0, \psi)_{H_\epsilon} + F(\phi - L_3^0, \psi), \\ \frac{1}{D_1} \left( \frac{\partial u}{\partial t}, P \right)_{H_\epsilon} &= -a_\epsilon(u - L_1^0, P) + \alpha_1 \left( (u - L_1^0) \mathcal{L}_\epsilon^{1/2}(\phi - L_3^0), \mathcal{L}_\epsilon^{1/2} P \right)_{H_\epsilon} \\ &\quad + G_1(u - L_1^0, \phi - L_3^0, P), \\ \frac{1}{D_2} \left( \frac{\partial v}{\partial t}, Q \right)_{H_\epsilon} &= -a_\epsilon(v - L_2^0, Q) - \alpha_2 \left( (v - L_2^0) \mathcal{L}_\epsilon^{1/2}(\phi - L_3^0), \mathcal{L}_\epsilon^{1/2} Q \right)_{H_\epsilon} \\ &\quad + G_2(v - L_2^0, \phi - L_3^0, Q), \end{aligned} \quad (3.41)$$

where, for appropriate functions  $p$ ,  $q$  and  $r$ , and for  $i = 1, 2$ ,

$$\begin{aligned} F(p, q) &= \left( \left( \frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2} \right) p_x, q \right)_{H_\epsilon} + \left( \frac{g_x}{g} p_x, yq_y + zq_z \right)_{H_\epsilon}, \\ G_i(p, q, r) &= - \left( \left( \frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2} \right) p_x, r \right)_{H_\epsilon} - \left( \frac{g_x}{g} p_x, yr_y + zr_z \right)_{H_\epsilon} \\ &\quad + (-1)^{i+1} \alpha_i \left( \left( \frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2} \right) pq_x, r \right)_{H_\epsilon} + (-1)^{i+1} \alpha_i \left( \frac{g_x}{g} pq_x, yr_y + zr_z \right)_{H_\epsilon}. \end{aligned} \quad (3.42)$$

Let

$$\psi^\epsilon = \phi_\epsilon - L_3^\epsilon - (\phi - L_3^0), \quad P^\epsilon = u_\epsilon - L_1^\epsilon - (u - L_1^0), \quad Q^\epsilon = v_\epsilon - L_2^\epsilon - (v - L_2^0). \quad (3.43)$$

Upon subtracting (3.41) from (3.40), we obtain that for any  $(\psi, P, Q) \in H_D^1(\Omega)^3$ ,

$$a_\epsilon(\psi^\epsilon, \psi) = \lambda \alpha_1(P^\epsilon, \psi)_{H_\epsilon} - \lambda \alpha_2(Q^\epsilon, \psi)_{H_\epsilon} + F(\phi - L_3^0, \psi), \quad (3.44)$$

$$\begin{aligned} \frac{1}{D_1} (\partial_t P^\epsilon, P)_{H_\epsilon} &= -a_\epsilon(P^\epsilon, P) + \alpha_1 \left( (u_\epsilon - L_1^\epsilon) \mathcal{L}_\epsilon^{\frac{1}{2}} \psi^\epsilon, \mathcal{L}_\epsilon^{\frac{1}{2}} P \right)_{H_\epsilon} \\ &\quad + \alpha_1 \left( P^\epsilon \mathcal{L}_\epsilon^{\frac{1}{2}} (\phi - L_3^0), \mathcal{L}_\epsilon^{\frac{1}{2}} P \right)_{H_\epsilon} - G_1(u - L_1^0, \phi - L_3^0, P), \end{aligned} \quad (3.45)$$

$$\begin{aligned} \frac{1}{D_2} (\partial_t Q^\epsilon, Q)_{H_\epsilon} &= -a_\epsilon(Q^\epsilon, Q) - \alpha_2 \left( (v_\epsilon - L_2^\epsilon) \mathcal{L}_\epsilon^{\frac{1}{2}} \psi^\epsilon, \mathcal{L}_\epsilon^{\frac{1}{2}} Q \right)_{H_\epsilon} \\ &\quad - \alpha_2 \left( Q^\epsilon \mathcal{L}_\epsilon^{\frac{1}{2}} (\phi - L_3^0), \mathcal{L}_\epsilon^{\frac{1}{2}} Q \right)_{H_\epsilon} - G_2(v - L_2^0, \phi - L_3^0, Q). \end{aligned} \quad (3.46)$$

For the above system, we have the following estimates.

**Lemma 3.9.** *There exists  $\epsilon_1 > 0$  such that, for any  $R > 0$ , there exists a constant  $K$  depending on  $R$  such that, for any  $0 < \epsilon \leq \epsilon_1$  and  $(u_0, v_0) \in \Sigma_\epsilon$  with  $\|(u_0, v_0)\|_{X_\epsilon \times X_\epsilon} \leq R$ , the following holds:*

$$\|P^\epsilon(t)\|_{H_\epsilon}^2 + \|Q^\epsilon(t)\|_{H_\epsilon}^2 + \|\psi^\epsilon(t)\|_{X_\epsilon}^2 + \int_0^t (\|P^\epsilon(s)\|_{X_\epsilon}^2 + \|Q^\epsilon(s)\|_{X_\epsilon}^2) ds \leq \epsilon K e^{Kt}, \quad t \geq 0,$$

where  $(\phi_\epsilon, u_\epsilon, v_\epsilon)$  is the solution of problem (3.3)-(3.5) with the initial condition  $(u_0, v_0)$ ,  $(\phi, u, v)$  is the solution of problem (3.7)-(3.9) with the initial condition  $(M(u_0), M(v_0))$ , and  $(\psi^\epsilon, P^\epsilon, Q^\epsilon)$  is given by (3.43).

*Proof.* It follows from (3.44)–(3.46) that

$$a_\epsilon(\psi^\epsilon, \psi^\epsilon) = \lambda\alpha_1(P^\epsilon, \psi^\epsilon)_{H_\epsilon} - \lambda\alpha_2(Q^\epsilon, \psi^\epsilon)_{H_\epsilon} + F(\phi - L_3^0, \psi^\epsilon), \quad (3.47)$$

$$\begin{aligned} \frac{1}{D_1}(\partial_t P^\epsilon, P^\epsilon)_{H_\epsilon} &= -a_\epsilon(P^\epsilon, P^\epsilon) + \alpha_1 \left( (u_\epsilon - L_1^\epsilon) \mathcal{L}_\epsilon^{\frac{1}{2}} \psi^\epsilon, \mathcal{L}_\epsilon^{\frac{1}{2}} P^\epsilon \right)_{H_\epsilon} \\ &\quad + \alpha_1 \left( P^\epsilon \mathcal{L}_\epsilon^{\frac{1}{2}} (\phi - L_3^0), \mathcal{L}_\epsilon^{\frac{1}{2}} P^\epsilon \right)_{H_\epsilon} - G_1(u - L_1^0, \phi - L_3^0, P^\epsilon), \end{aligned} \quad (3.48)$$

$$\begin{aligned} \frac{1}{D_2}(\partial_t Q^\epsilon, Q^\epsilon)_{H_\epsilon} &= -a_\epsilon(Q^\epsilon, Q^\epsilon) - \alpha_2 \left( (v_\epsilon - L_2^\epsilon) \mathcal{L}_\epsilon^{\frac{1}{2}} \psi^\epsilon, \mathcal{L}_\epsilon^{\frac{1}{2}} Q^\epsilon \right)_{H_\epsilon} \\ &\quad - \alpha_2 \left( Q^\epsilon \mathcal{L}_\epsilon^{\frac{1}{2}} (\phi - L_3^0), \mathcal{L}_\epsilon^{\frac{1}{2}} Q^\epsilon \right)_{H_\epsilon} - G_2(v - L_2^0, \phi - L_3^0, Q^\epsilon). \end{aligned} \quad (3.49)$$

Next, we estimate each term on the right-hand sides of (3.47)–(3.49). The first two terms on the right-hand side of (3.47) are bounded by:

$$\begin{aligned} \lambda\alpha_1 |(P^\epsilon, \psi^\epsilon)_{H_\epsilon}| + \lambda\alpha_2 |(Q^\epsilon, \psi^\epsilon)_{H_\epsilon}| &\leq C(\|P^\epsilon\|_{H_\epsilon} + \|Q^\epsilon\|_{H_\epsilon}) \|\psi^\epsilon\|_{H_\epsilon} \\ &\leq C(\|P^\epsilon\|_{H_\epsilon}^2 + \|Q^\epsilon\|_{H_\epsilon}^2) + \frac{1}{4}a_\epsilon(\psi^\epsilon, \psi^\epsilon). \end{aligned} \quad (3.50)$$

By (3.11) we find that  $g$  satisfies

$$\left| \frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2} \right| \leq C\epsilon.$$

Then, by Lemma 3.5, the first term in  $F(\phi - L_3^0, \psi^\epsilon)$  on the right-hand side of (3.47) is less than

$$\left| \left( \left( \frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2} \right) (\phi - L_3^0)_x, \psi^\epsilon \right)_{H_\epsilon} \right| \leq C\epsilon \|\psi^\epsilon\|_{H_\epsilon}^2 \leq C\epsilon^2 + \frac{1}{4}a_\epsilon(\psi^\epsilon, \psi^\epsilon). \quad (3.51)$$

It follows from (3.30) and Lemma 3.5 that the second term in  $F(\phi - L_3^0, \psi^\epsilon)$  on the right-hand side of (3.47) is bounded by, for  $t \geq 0$ ,

$$\begin{aligned} &\left| \left( \frac{g_x}{g} (\phi - L_3^0)_x, y \partial_y (\phi_\epsilon - L_3^\epsilon) + z \partial_z (\phi_\epsilon - L_3^\epsilon) \right)_{H_\epsilon} \right| \\ &\leq C(\|\partial_y \phi_\epsilon\|_{H_\epsilon} + \|\partial_z \phi_\epsilon\|_{H_\epsilon} + \|\partial_y L_3^\epsilon\|_{H_\epsilon} + \|\partial_z L_3^\epsilon\|_{H_\epsilon}) \leq C\epsilon. \end{aligned} \quad (3.52)$$

By (3.47) and (3.50)–(3.52), we obtain, for all  $t \geq 0$ ,

$$\|\psi^\epsilon(t)\|_{X_\epsilon}^2 \leq C a_\epsilon(\psi^\epsilon, \psi^\epsilon) \leq C(\|P^\epsilon\|_{H_\epsilon}^2 + \|Q^\epsilon\|_{H_\epsilon}^2) + C\epsilon. \quad (3.53)$$

We now deal with the right-hand side of (3.48). By (3.53), the second term on the right-hand side of (3.48) is less than

$$\alpha_1 \left| \left( (u_\epsilon - L_1) \mathcal{L}_\epsilon^{\frac{1}{2}} \psi^\epsilon, \mathcal{L}_\epsilon^{\frac{1}{2}} P^\epsilon \right)_{H_\epsilon} \right| \leq C(\|\psi^\epsilon\|_{X_\epsilon}^2 + \|P^\epsilon\|_{X_\epsilon}^2) \leq C(\|P^\epsilon\|_{X_\epsilon}^2 + \|Q^\epsilon\|_{X_\epsilon}^2) + C\epsilon. \quad (3.54)$$

Since the functions  $\phi$  and  $L_3^0$  depend on  $x \in (0, 1)$  only, we have

$$\mathcal{L}_\epsilon^{\frac{1}{2}}\phi = J^T \nabla \phi = (\partial_x \phi, 0, 0)^\tau, \quad \mathcal{L}_\epsilon^{\frac{1}{2}}L_3^0 = J^T \nabla L_3^0 = (\partial_x L_3^0, 0, 0)^\tau,$$

which, along with Lemma 3.5 and the first equation of (3.7), implies that, for all  $t \geq 0$ ,

$$\|\mathcal{L}_\epsilon^{\frac{1}{2}}\phi\|_\infty = \|\partial_x \phi\|_\infty \leq C\|\partial_x \phi\|_{H^1} \leq C\|\phi\|_{H^2} \leq C. \quad (3.55)$$

By (3.55), the third term on the right-hand side of (3.48) is bounded by

$$\begin{aligned} \alpha_1 \left| \left( P^\epsilon \mathcal{L}_\epsilon^{\frac{1}{2}}(\phi - L_3^0), \mathcal{L}_\epsilon^{\frac{1}{2}}P^\epsilon \right)_{H_\epsilon} \right| &\leq \alpha_1 \left( \|\mathcal{L}_\epsilon^{\frac{1}{2}}\phi\|_\infty + \|\mathcal{L}_\epsilon^{\frac{1}{2}}L_3^0\|_\infty \right) \|P^\epsilon\|_{H_\epsilon} \|\mathcal{L}_\epsilon^{\frac{1}{2}}P^\epsilon\|_{H_\epsilon} \\ &\leq C\|P^\epsilon\|_{H_\epsilon}^2 + \frac{1}{4}a_\epsilon(P^\epsilon, P^\epsilon). \end{aligned} \quad (3.56)$$

Note that the term  $G_1$  on the right-hand side of (3.48) can be estimated in a similar manner as (3.50)-(3.52). Therefore, it follows from (3.48) and (3.53)-(3.56) that, for  $t \geq 0$ ,

$$\frac{d}{dt}\|P^\epsilon\|_{H_\epsilon}^2 + \|P^\epsilon\|_{X_\epsilon}^2 \leq C(\|P^\epsilon\|_{H_\epsilon}^2 + \|Q^\epsilon\|_{H_\epsilon}^2) + C\epsilon. \quad (3.57)$$

Similarly,  $Q^\epsilon$  satisfies, for  $t \geq 0$ ,

$$\frac{d}{dt}\|Q^\epsilon\|_{H_\epsilon}^2 + \|Q^\epsilon\|_{X_\epsilon}^2 \leq C(\|P^\epsilon\|_{H_\epsilon}^2 + \|Q^\epsilon\|_{H_\epsilon}^2) + C\epsilon. \quad (3.58)$$

Then, it follows from (3.57)-(3.58) that, for  $t \geq 0$ ,

$$\frac{d}{dt}(\|P^\epsilon\|_{H_\epsilon}^2 + \|Q^\epsilon\|_{H_\epsilon}^2) + \|P^\epsilon\|_{X_\epsilon}^2 + \|Q^\epsilon\|_{X_\epsilon}^2 \leq C(\|P^\epsilon\|_{H_\epsilon}^2 + \|Q^\epsilon\|_{H_\epsilon}^2) + C\epsilon. \quad (3.59)$$

By Gronwall's lemma, we get

$$\begin{aligned} \|P^\epsilon(t)\|_{H_\epsilon}^2 + \|Q^\epsilon(t)\|_{H_\epsilon}^2 &\leq e^{Ct}(\|P^\epsilon(0)\|_{H_\epsilon}^2 + \|Q^\epsilon(0)\|_{H_\epsilon}^2) + \epsilon e^{Ct} \\ &\leq C e^{Ct}(\|u_0 - M(u_0)\|_{H_\epsilon}^2 + \|L_1^\epsilon - L_1^0\|_{H_\epsilon}^2 + \|v_0 - M(v_0)\|_{H_\epsilon}^2 + \|L_2^\epsilon - L_2^0\|_{H_\epsilon}^2) + \epsilon e^{Ct}. \end{aligned} \quad (3.60)$$

By (3.2) we see that  $L_1, L_2 \in W^{1,\infty}(\Omega_\epsilon)$ , and hence, for  $k = 1, 2$ ,

$$\|L_k^\epsilon - L_k^0\|_{H_\epsilon}^2 = \left\| \int_0^1 \left( yg \frac{\partial L_k}{\partial Y}(x, sgy, sgz) + zg \frac{\partial L_k}{\partial Z}(x, sgy, sgz) \right) ds \right\|_{H_\epsilon}^2 \leq C\epsilon^2. \quad (3.61)$$

From (3.60)-(3.61) and Lemma 3.8, we find that

$$\|P^\epsilon(t)\|_{H_\epsilon}^2 + \|Q^\epsilon(t)\|_{H_\epsilon}^2 \leq \epsilon(C + 1)e^{Ct}. \quad (3.62)$$

Integrating (3.59) between 0 and  $t$ , by (3.62) we conclude Lemma 3.9.  $\square$

Next, we improve the uniform estimates in  $\epsilon$  given in Lemma 3.9.

**Lemma 3.10.** *There exists  $\epsilon_1 > 0$  such that, for any  $R > 0$ , there exists a constant  $K$  depending on  $R$  such that, for any  $0 < \epsilon \leq \epsilon_1$  and  $(u_0, v_0) \in \Sigma_\epsilon$  with  $\|(u_0, v_0)\|_{X_\epsilon \times X_\epsilon} \leq R$ , the following holds:*

$$t^2 \left( \left\| \frac{\partial P^\epsilon}{\partial t} \right\|_{H_\epsilon}^2 + \left\| \frac{\partial Q^\epsilon}{\partial t} \right\|_{H_\epsilon}^2 \right) + t \left( \|P^\epsilon\|_{X_\epsilon}^2 + \|Q^\epsilon\|_{X_\epsilon}^2 \right) \leq \sqrt{\epsilon} K e^{Kt}, \quad t \geq 0,$$

where  $(\psi^\epsilon, P^\epsilon, Q^\epsilon)$  is given by (3.43),  $(\phi_\epsilon, u_\epsilon, v_\epsilon)$  is the solution of problem (3.3)-(3.5) with the initial condition  $(u_0, v_0)$ , and  $(\phi, u, v)$  is the solution of problem (3.7)-(3.9) with the initial condition  $(M(u_0), M(v_0))$ .

*Proof.* Denote by

$$\tilde{P}^\epsilon = \frac{\partial P^\epsilon}{\partial t}, \quad \tilde{Q}^\epsilon = \frac{\partial Q^\epsilon}{\partial t}, \quad \tilde{\psi}^\epsilon = \frac{\partial \psi^\epsilon}{\partial t}. \quad (3.63)$$

Differentiating systems (3.44)–(3.46) with respect to  $t$ , multiplying the resulting systems by  $t$ , replacing  $\psi$ ,  $P$  and  $Q$  by  $t\tilde{\psi}$ ,  $t\tilde{P}$  and  $t\tilde{Q}$ , respectively, we obtain

$$a_\epsilon(t\tilde{\psi}^\epsilon, t\tilde{\psi}^\epsilon) = \lambda\alpha_1(t\tilde{P}^\epsilon, t\tilde{\psi}^\epsilon)_{H_\epsilon} - \lambda\alpha_2(t\tilde{Q}^\epsilon, t\tilde{\psi}^\epsilon)_{H_\epsilon} + tF(\phi_t, t\tilde{\psi}^\epsilon), \quad (3.64)$$

$$\begin{aligned} \frac{1}{2D_1} \frac{d}{dt} \|t\tilde{P}^\epsilon\|_{H_\epsilon}^2 + a_\epsilon(t\tilde{P}^\epsilon, t\tilde{P}^\epsilon) &= \alpha_1 t \left( \partial_t u_\epsilon \mathcal{L}_\epsilon^{\frac{1}{2}} \psi^\epsilon, \mathcal{L}_\epsilon^{\frac{1}{2}} t\tilde{P}^\epsilon \right)_{H_\epsilon} + \alpha_1 t \left( (u_\epsilon - L_1^\epsilon) \mathcal{L}_\epsilon^{\frac{1}{2}} \tilde{\psi}^\epsilon, \mathcal{L}_\epsilon^{\frac{1}{2}} t\tilde{P}^\epsilon \right)_{H_\epsilon} \\ &+ \alpha_1 t \left( \tilde{P}^\epsilon \mathcal{L}_\epsilon^{\frac{1}{2}} (\phi - L_3^0), \mathcal{L}_\epsilon^{\frac{1}{2}} t\tilde{P}^\epsilon \right)_{H_\epsilon} + \alpha_1 t \left( P^\epsilon \mathcal{L}_\epsilon^{\frac{1}{2}} \phi_t, \mathcal{L}_\epsilon^{\frac{1}{2}} t\tilde{P}^\epsilon \right)_{H_\epsilon} \\ &- \alpha_1 t \left( \left( \frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2} \right) (u - L_1^0) \phi_{tx}, t\tilde{P}^\epsilon \right)_{H_\epsilon} \\ &- \alpha_1 t \left( \frac{g_x}{g} (u - L_1^0) \phi_{tx}, ty \partial_y \tilde{P}^\epsilon + tz \partial_z \tilde{P}^\epsilon \right)_{H_\epsilon} \\ &- tG_1(u_t, \phi - L_3^0, t\tilde{P}^\epsilon) + \frac{1}{D_1} (\tilde{P}^\epsilon, t\tilde{P}^\epsilon)_{H_\epsilon}, \end{aligned} \quad (3.65)$$

$$\begin{aligned} \frac{1}{2D_2} \frac{d}{dt} \|t\tilde{Q}^\epsilon\|_{H_\epsilon}^2 + a_\epsilon(t\tilde{Q}^\epsilon, t\tilde{Q}^\epsilon) &= -\alpha_2 t \left( \partial_t v_\epsilon \mathcal{L}_\epsilon^{\frac{1}{2}} \psi^\epsilon, \mathcal{L}_\epsilon^{\frac{1}{2}} t\tilde{Q}^\epsilon \right)_{H_\epsilon} - \alpha_2 t \left( (v_\epsilon - L_2^\epsilon) \mathcal{L}_\epsilon^{\frac{1}{2}} \tilde{\psi}^\epsilon, \mathcal{L}_\epsilon^{\frac{1}{2}} t\tilde{Q}^\epsilon \right)_{H_\epsilon} \\ &- \alpha_2 t \left( \tilde{Q}^\epsilon \mathcal{L}_\epsilon^{\frac{1}{2}} (\phi - L_3^0), \mathcal{L}_\epsilon^{\frac{1}{2}} t\tilde{Q}^\epsilon \right)_{H_\epsilon} - \alpha_2 t \left( Q^\epsilon \mathcal{L}_\epsilon^{\frac{1}{2}} \phi_t, \mathcal{L}_\epsilon^{\frac{1}{2}} t\tilde{Q}^\epsilon \right)_{H_\epsilon} \\ &+ \alpha_2 t \left( \left( \frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2} \right) (v - L_2^0) \phi_{tx}, t\tilde{Q}^\epsilon \right)_{H_\epsilon} \\ &+ \alpha_2 t \left( \frac{g_x}{g} (v - L_2^0) \phi_{tx}, ty \partial_y \tilde{Q}^\epsilon + tz \partial_z \tilde{Q}^\epsilon \right)_{H_\epsilon} \\ &- tG_2(v_t, \phi - L_3^0, t\tilde{Q}^\epsilon) + \frac{1}{D_2} (\tilde{Q}^\epsilon, t\tilde{Q}^\epsilon)_{H_\epsilon}. \end{aligned} \quad (3.66)$$



We now estimate every term involved in the above system. Note that (3.64) implies that

$$\|\mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{\psi}^\epsilon\|_{H_\epsilon}^2 \leq C \left( \|t \tilde{P}^\epsilon\|_{H_\epsilon}^2 + \|t \tilde{Q}^\epsilon\|_{H_\epsilon}^2 \right) + C \epsilon^2 t^2. \quad (3.67)$$

By (3.30) and Lemma 3.9, we see that the first term on the right-hand side of (3.65) is bounded by

$$\begin{aligned} |\alpha_1 t \left( \partial_t u_\epsilon \mathcal{L}_\epsilon^{\frac{1}{2}} \psi^\epsilon, \mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon \right)_{H_\epsilon}| &\leq C t \|\partial_t u_\epsilon\|_6 \|\mathcal{L}_\epsilon^{\frac{1}{2}} \psi^\epsilon\|_3 \|\mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon\|_2 \\ &\leq C t \|\partial_t u_\epsilon\|_{H^1} \|\mathcal{L}_\epsilon^{\frac{1}{2}} \psi^\epsilon\|_2^{\frac{1}{2}} \|\mathcal{L}_\epsilon^{\frac{1}{2}} \psi^\epsilon\|_{H^1}^{\frac{1}{2}} \|\mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon\|_2 \\ &\leq \frac{1}{32} \|\mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon\|_{H_\epsilon}^2 + C t^2 \|\mathcal{L}_\epsilon^{\frac{1}{2}} \partial_t u_\epsilon\|_{H_\epsilon}^2 \|\mathcal{L}_\epsilon^{\frac{1}{2}} \psi^\epsilon\|_{H_\epsilon} \|\mathcal{L}_\epsilon \psi^\epsilon\|_{H_\epsilon} \\ &\leq \frac{1}{32} \|\mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon\|_{H_\epsilon}^2 + \sqrt{\epsilon} C e^{Ct} \|t \mathcal{L}_\epsilon^{\frac{1}{2}} \partial_t u_\epsilon\|_{H_\epsilon}^2. \end{aligned} \quad (3.68)$$

By (3.67), the second term on the right-hand side of (3.65) is less than

$$\alpha_1 t \left( (u_\epsilon - L_1^\epsilon) \mathcal{L}_\epsilon^{\frac{1}{2}} \tilde{\psi}^\epsilon, \mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon \right)_{H_\epsilon} \leq \frac{1}{32} \|\mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon\|_{H_\epsilon}^2 + C \left( \|t \tilde{P}^\epsilon\|_{H_\epsilon}^2 + \|t \tilde{Q}^\epsilon\|_{H_\epsilon}^2 \right) + \epsilon^2 C t^2. \quad (3.69)$$

By Lemma 3.7, the fourth term on the right-hand side of (3.65) is bounded by

$$C \|t \mathcal{L}_\epsilon^{\frac{1}{2}} \phi_t\|_\infty \|P^\epsilon\|_2 \|\mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon\|_2 \leq \frac{1}{32} \|\mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon\|_{H_\epsilon}^2 + C \|t \phi_t\|_{H^2}^2 \|P^\epsilon\|_{H_\epsilon}^2 \leq \frac{1}{32} \|\mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon\|_{H_\epsilon}^2 + \epsilon C e^{Ct}. \quad (3.70)$$

Other terms on the right-hand side of (3.65) can be estimated in a similar way as the proof of Lemma 3.9. Therefore, by (3.65), (3.68)-(3.70) and the estimates for other terms, we have

$$\begin{aligned} \frac{1}{2D_1} \frac{d}{dt} \|t \tilde{P}^\epsilon\|_{H_\epsilon}^2 + \frac{3}{4} \|\mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon\|_{H_\epsilon}^2 &\leq C \left( \|t \tilde{P}^\epsilon\|_{H_\epsilon}^2 + \|t \tilde{Q}^\epsilon\|_{H_\epsilon}^2 \right) + \epsilon C e^{Ct} + \epsilon^2 C (\|tu_t\|_{H^1}^2 + \|tv_t\|_{H^1}^2) \\ &\quad + \sqrt{\epsilon} C e^{Ct} \|t \mathcal{L}_\epsilon^{\frac{1}{2}} \partial_t u_\epsilon\|_{H_\epsilon}^2 + \frac{1}{D_1} (\tilde{P}^\epsilon, t \tilde{P}^\epsilon)_{H_\epsilon}. \end{aligned} \quad (3.71)$$

Next, we deal with the last term on the right-hand side of the above inequality. Replacing  $P$  in (3.45) by  $t \partial_t P^\epsilon = t \tilde{P}^\epsilon$ , we get

$$\begin{aligned} \frac{1}{D_1} (\tilde{P}^\epsilon, t \tilde{P}^\epsilon)_{H_\epsilon} + \frac{1}{2} t \frac{d}{dt} \|\mathcal{L}_\epsilon^{\frac{1}{2}} P^\epsilon\|_{H_\epsilon}^2 &= \alpha_1 \left( (u_\epsilon - L_1^\epsilon) \mathcal{L}_\epsilon^{\frac{1}{2}} \psi^\epsilon, \mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon \right)_{H_\epsilon} \\ &\quad + \alpha_1 \left( P^\epsilon \mathcal{L}_\epsilon^{\frac{1}{2}} (\phi - L_3^0), \mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon \right)_{H_\epsilon} - G_1(u - L_1^0, \phi - L_3^0, t \tilde{P}^\epsilon). \end{aligned}$$

Using Lemma 3.9 and proceeding as before, we obtain from the above that

$$\frac{1}{D_1} (\tilde{P}^\epsilon, t \tilde{P}^\epsilon)_{H_\epsilon} + \frac{1}{2} t \frac{d}{dt} \|\mathcal{L}_\epsilon^{\frac{1}{2}} P^\epsilon\|_{H_\epsilon}^2 \leq \frac{1}{4} \|\mathcal{L}_\epsilon^{\frac{1}{2}} t \tilde{P}^\epsilon\|_{H_\epsilon}^2 + \epsilon C e^{Ct} + \epsilon^2 C. \quad (3.72)$$

Then it follows from (3.71)-(3.72) that

$$\begin{aligned} & \frac{1}{2D_1} \frac{d}{dt} \|t\tilde{P}^\epsilon\|_{H_\epsilon}^2 + \frac{1}{2} \|\mathcal{L}_\epsilon^{\frac{1}{2}} t\tilde{P}^\epsilon\|_{H_\epsilon}^2 + \frac{1}{2} t \frac{d}{dt} \|\mathcal{L}_\epsilon^{\frac{1}{2}} P^\epsilon\|_{H_\epsilon}^2 \\ & \leq C \left( \|t\tilde{P}^\epsilon\|_{H_\epsilon}^2 + \|t\tilde{Q}^\epsilon\|_{H_\epsilon}^2 \right) + \epsilon C e^{Ct} \\ & \quad + \epsilon^2 C (\|tu_t\|_{H^1}^2 + \|tv_t\|_{H^1}^2) + \sqrt{\epsilon} C e^{Ct} \|t\mathcal{L}_\epsilon^{\frac{1}{2}} \partial_t u_\epsilon\|_{H_\epsilon}^2, \end{aligned}$$

which implies that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \frac{1}{D_1} \|t\tilde{P}^\epsilon\|_{H_\epsilon}^2 + t \|\mathcal{L}_\epsilon^{\frac{1}{2}} P^\epsilon\|_{H_\epsilon}^2 \right) & \leq \frac{1}{2} \|\mathcal{L}_\epsilon^{\frac{1}{2}} P^\epsilon\|_{H_\epsilon}^2 + C \left( \|t\tilde{P}^\epsilon\|_{H_\epsilon}^2 + \|t\tilde{Q}^\epsilon\|_{H_\epsilon}^2 \right) + \epsilon C e^{Ct} \\ & \quad + \epsilon^2 C (\|tu_t\|_{H^1}^2 + \|tv_t\|_{H^1}^2) + \sqrt{\epsilon} C e^{Ct} \|t\mathcal{L}_\epsilon^{\frac{1}{2}} \partial_t u_\epsilon\|_{H_\epsilon}^2. \end{aligned} \quad (3.73)$$

Similarly, by equation (3.66), we can show that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \frac{1}{D_2} \|t\tilde{Q}^\epsilon\|_{H_\epsilon}^2 + t \|\mathcal{L}_\epsilon^{\frac{1}{2}} Q^\epsilon\|_{H_\epsilon}^2 \right) & \leq \frac{1}{2} \|\mathcal{L}_\epsilon^{\frac{1}{2}} Q^\epsilon\|_{H_\epsilon}^2 + C \left( \|t\tilde{P}^\epsilon\|_{H_\epsilon}^2 + \|t\tilde{Q}^\epsilon\|_{H_\epsilon}^2 \right) + \epsilon C e^{Ct} \\ & \quad + \epsilon^2 C (\|tu_t\|_{H^1}^2 + \|tv_t\|_{H^1}^2) + \sqrt{\epsilon} C e^{Ct} \|t\mathcal{L}_\epsilon^{\frac{1}{2}} \partial_t v_\epsilon\|_{H_\epsilon}^2. \end{aligned} \quad (3.74)$$

By (3.73)-(3.74) we find that

$$\begin{aligned} & \frac{d}{dt} \left( \frac{1}{D_1} \|t\tilde{P}^\epsilon\|_{H_\epsilon}^2 + t \|\mathcal{L}_\epsilon^{\frac{1}{2}} P^\epsilon\|_{H_\epsilon}^2 + \frac{1}{D_2} \|t\tilde{Q}^\epsilon\|_{H_\epsilon}^2 + t \|\mathcal{L}_\epsilon^{\frac{1}{2}} Q^\epsilon\|_{H_\epsilon}^2 \right) \\ & \leq C \left( \frac{1}{D_1} \|t\tilde{P}^\epsilon\|_{H_\epsilon}^2 + t \|\mathcal{L}_\epsilon^{\frac{1}{2}} P^\epsilon\|_{H_\epsilon}^2 + \frac{1}{D_2} \|t\tilde{Q}^\epsilon\|_{H_\epsilon}^2 + t \|\mathcal{L}_\epsilon^{\frac{1}{2}} Q^\epsilon\|_{H_\epsilon}^2 \right) + \|\mathcal{L}_\epsilon^{\frac{1}{2}} P^\epsilon\|_{H_\epsilon}^2 + \|\mathcal{L}_\epsilon^{\frac{1}{2}} Q^\epsilon\|_{H_\epsilon}^2 \\ & \quad + \epsilon C e^{Ct} + \epsilon^2 C (\|tu_t\|_{H^1}^2 + \|tv_t\|_{H^1}^2) + \sqrt{\epsilon} C e^{Ct} \left( \|t\mathcal{L}_\epsilon^{\frac{1}{2}} \partial_t u_\epsilon\|_{H_\epsilon}^2 + \|t\mathcal{L}_\epsilon^{\frac{1}{2}} \partial_t v_\epsilon\|_{H_\epsilon}^2 \right), \end{aligned}$$

which, along with Gronwall's lemma and Lemmas 3.6, 3.7 and 3.9, implies Lemma 3.10.  $\square$

Let  $(c_1^\epsilon, c_2^\epsilon, \Phi^\epsilon)$  be the solutions of problem (2.4)-(2.5) with initial datum  $(c_{1,0}, c_{2,0})$ , and  $(c_1, c_2, \Phi)$  be the solutions of problem (2.6)-(2.7) with initial datum  $(M(c_{1,0}), M(c_{2,0}))$ . Then as an immediate consequence of Lemma 3.10, we find the following estimates which are essential to prove the upper semi-continuity of the global attractors.

**Lemma 3.11.** *There exists  $\epsilon_1 > 0$  such that, for any  $R > 0$ , there exists a constant  $K$  depending on  $R$  such that, for any  $0 < \epsilon \leq \epsilon_1$  and  $(c_{1,0}, c_{2,0}) \in \tilde{\Sigma}$  with  $\|(c_{1,0}, c_{2,0})\|_{X_\epsilon \times X_\epsilon} \leq R$ , the following holds:*

$$(\|c_1^\epsilon(t) - c_1(t)\|_{X_\epsilon}^2 + \|c_2^\epsilon(t) - c_2(t)\|_{X_\epsilon}^2) \leq \sqrt{\epsilon} K e^{Kt}, \quad t \geq 1.$$

We are now in a position to prove the upper semi-continuity of global attractors.

**Proof of Theorem 2.3.** Let  $T^\epsilon(t)_{t \geq 0}$  and  $T^0(t)_{t \geq 0}$  be the solution operators of problem (2.4)-(2.5) and problem (2.6)-(2.7), respectively. Then it follows from Proposition 3.4 that there is a constant  $R > 0$  (independent of  $\epsilon$ ) such that

$$\|(c_1, c_2)\|_{X_\epsilon \times X_\epsilon} \leq R, \text{ for all } (c_1, c_2) \in \mathcal{A}_\epsilon.$$

For the given  $\eta > 0$ , since  $\mathcal{A}_0$  is the global attractor of  $T^0(t)$ , there exists  $\tau_0 = \tau_0(\eta, R) \geq 1$  such that, for any  $t \geq \tau_0$ ,

$$\inf_{z_0 \in \mathcal{A}_0} \|T^0(t)(Mz) - z_0\|_{X_\epsilon \times X_\epsilon} \leq \frac{\eta}{2},$$

for any  $z = (c_1, c_2) \in \mathcal{A}_\epsilon$ . On the other hand, by Lemma 3.11 we find that

$$\|T^\epsilon(\tau_0)z - T^0(\tau_0)(Mz)\|_{X_\epsilon \times X_\epsilon} \leq \epsilon^{\frac{1}{4}} K(R) e^{K(R)\tau_0},$$

for some constant  $K(R)$ . Therefore, we obtain that, for any  $z = (c_1, c_2) \in \mathcal{A}_\epsilon$ :

$$\inf_{z_0 \in \mathcal{A}_0} \|T^\epsilon(\tau_0)z - z_0\|_{X_\epsilon \times X_\epsilon} \leq \frac{\eta}{2} + \epsilon^{\frac{1}{4}} K(R) e^{K(R)\tau_0},$$

which implies that, for  $\epsilon > 0$  small enough:

$$\text{dist}_{X_\epsilon \times X_\epsilon}(T^\epsilon(\tau_0)\mathcal{A}_\epsilon, \mathcal{A}_0) \leq \eta.$$

The proof is completed since  $T^\epsilon(\tau_0)\mathcal{A}_\epsilon = \mathcal{A}_\epsilon$ .

## 4 Steady-states for the one-dimensional limiting PNP system

Many mathematical works have been done on the existence, uniqueness and qualitative properties of boundary value problems even for high dimensional systems and algorithms have been developed toward numerical approximations (see, e.g. [15, 16, 25, 17]). Under the assumption that  $\mu \ll 1$ , the problem can be viewed as a singularly perturbed system. Typical solutions of singularly perturbed systems exhibit different time scales; for example, boundary and internal layers (inner solutions) evolve at fast pace and regular layers (outer solutions) vary slowly. For the boundary value problem (2.8) and (2.9), there are two boundary layers one at each end. Physically, near boundaries  $x = 0$  and  $x = 1$ , the potential function  $\phi(x)$  and the concentration functions  $c_1(x)$  and  $c_2(x)$  exhibit a large gradient or a sharp change. In [2], for  $\alpha_1 = \alpha_2 = 1$ , the boundary value problem for the direct (with  $h(x) = 1$  in (2.8)) one-dimensional PNP system was studied using the method of matched asymptotic expansions as well as numerical simulations, which provide a good quantitative and qualitative understanding of the problem. In [23], geometric singular perturbation theory (see, e.g. [6, 18, 20, 22]) was applied to the study of this singular boundary

value problem. The treatment for the limiting one-dimensional PNP system (2.8) carrying the geometric information of the three-dimensional channel follows that in [23].

It is convenient to study an equivalent connecting problem to the boundary value problem (2.8) and (2.9). Let  $B_L$  and  $B_R$  be the subsets of  $\mathbb{R}^7$  defined, respectively, by

$$\begin{aligned} B_L &= \{\phi = \phi_0, v = -h(0)(\alpha_1 l_1 - \alpha_2 l_2), w = \alpha_1^2 l_1 + \alpha_2^2 l_2, \tau = 0\}, \\ B_R &= \{\phi = 0, v = -h(1)(\alpha_1 r_1 - \alpha_2 r_2), w = \alpha_1^2 r_1 + \alpha_2^2 r_2, \tau = 1\}. \end{aligned} \quad (4.1)$$

The boundary value problem is then equivalent to the following *connecting problem*: finding a solution of (2.10) from  $B_L$  to  $B_R$ .

For  $\mu > 0$ , let  $M_L^\mu$  be the union of all forward orbits of (2.10) starting from  $B_L$  and let  $M_R^\mu$  be the union of all backward orbits starting from  $B_R$ . To obtain the existence and (local) uniqueness of a solution for the connecting problem, it thus suffices to show  $M_L^\mu$  and  $M_R^\mu$  intersect transversally. The intersection is exactly the orbit of a solution of the boundary value problem, and the transversality implies the local uniqueness. The strategy is to obtain a singular orbit and track the evolution of  $M_L^\mu$  and  $M_R^\mu$  along the singular orbit. As discussed in the introduction, a singular orbit will be a union of orbits of subsystems of (2.10) with different time scales.

The boundary layers will be two orbits of (2.14): one from  $B_L$  to  $\mathcal{Z}_0$  in forward time along the stable manifold of  $\mathcal{Z}_0$  and the other from  $B_R$  to  $\mathcal{Z}_0$  in backward time along the unstable manifold of  $\mathcal{Z}_0$ . The two boundary layers will be connected by a regular layer on  $\mathcal{Z}_0$ , which is an orbit of a limiting system of (2.10). The next two subsections are devoted to the study of boundary layers and regular layers.

## 4.1 Fast dynamics and boundary layers

We start with the study of boundary layers governed by system (2.14). This system has many invariant structures that are useful for characterizing the global dynamics.

The slow manifold  $\mathcal{Z}_0 = \{u = v = 0\}$  consisting of entirely equilibria of system (2.14) is a 5-dimensional manifold of the phase space  $\mathbb{R}^7$ . For each equilibrium  $z = (\phi, 0, 0, w, J_1, J_2, \tau) \in \mathcal{Z}_0$ , the linearization of system (2.14) has five zero eigenvalues corresponding to the dimension of  $\mathcal{Z}_0$ , and two eigenvalues in directions normal to  $\mathcal{Z}_0$ . The latter two eigenvalues and their associated eigenvectors are given by

$$\lambda_\pm = \pm\sqrt{w} \quad \text{and} \quad n_\pm = ((\pm\sqrt{w})^{-1}, 1, \pm\sqrt{w}, \pm(\alpha_2 - \alpha_1)\sqrt{w}, 0, 0, 0)^\tau. \quad (4.2)$$

Thus, every equilibrium has a one-dimensional stable manifold and a one-dimensional unstable manifold. The global configurations of the stable and unstable manifolds will be needed for the

boundary layer behavior. For any constants  $J_1^*$ ,  $J_2^*$  and  $\tau^*$ , the set  $\mathcal{N} = \{J_1 = J_1^*, J_2 = J_2^*, \tau = \tau^*\}$  is a 4-dimensional invariant subspace of the phase space  $\mathbb{R}^7$ .

Surprisingly, system (2.14) possesses a complete set of integrals with which the dynamics can be fully analyzed; in particular, the stable and unstable manifolds can be characterized and the behavior of boundary layers can be described in detail.

**Proposition 4.1.** (i) System (2.14) has a complete set of six integrals given by

$$H_1 = w - \frac{\alpha_2 - \alpha_1}{h(\tau)}v - \frac{\alpha_1\alpha_2}{2h^2(\tau)}u^2, \quad H_2 = \phi - \frac{\ln|\alpha_1 v/h(\tau) + w|}{\alpha_2},$$

$$H_3 = \phi + \frac{\ln|\alpha_2 v/h(\tau) - w|}{\alpha_1}, \quad H_4 = J_1, \quad H_5 = J_2 \quad \text{and} \quad H_6 = \tau,$$

where the argument of  $H_i$ 's is  $(\phi, u, v, w, J_1, J_2, \tau)$ .

(ii) The stable and unstable manifolds  $W^s(\mathcal{Z}_0)$  and  $W^u(\mathcal{Z}_0)$  of  $\mathcal{Z}_0$  are characterized as follows:

$$W^s(\mathcal{Z}_0) = \cup\{W^s(z^*) : z^* \in \mathcal{Z}_0\} \quad \text{and} \quad W^u(\mathcal{Z}_0) = \cup\{W^u(z^*) : z^* \in \mathcal{Z}_0\}$$

and, for  $z^* = (\phi^*, 0, 0, w^*, J_1^*, J_2^*, \tau^*) \in \mathcal{Z}_0$ , a point  $z = (\phi, u, v, w, J_1, J_2, \tau) \in W^s(z^*) \cup W^u(z^*)$  if and only if

$$H_1(z) = w^*, \quad H_2(z) = \phi^* - \frac{\ln w^*}{\alpha_2}, \quad H_3(z) = \phi^* + \frac{\ln w^*}{\alpha_1},$$

$$J_1 = J_1^*, \quad J_2 = J_2^*, \quad \tau = \tau^*.$$

(iii) The stable manifold  $W^s(\mathcal{Z}_0)$  intersects  $B_L$  transversally at points with

$$u = -[\text{sgn}(\alpha_2 l_2 - \alpha_1 l_1)]\sqrt{2}h(0)\sqrt{l_1 + l_2 - \frac{(\alpha_1 + \alpha_2)(\alpha_1 l_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}}(\alpha_2 l_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}}}{\alpha_1 \alpha_2}} \quad (4.3)$$

and arbitrary  $J_1$  and  $J_2$ , where  $\text{sgn}$  is the sign function. The unstable manifold  $W^u(\mathcal{Z}_0)$  intersects  $B_R$  transversally at points with

$$u = [\text{sgn}(\alpha_2 r_2 - \alpha_1 r_1)]\sqrt{2}h(1)\sqrt{r_1 + r_2 - \frac{(\alpha_1 + \alpha_2)(\alpha_1 r_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}}(\alpha_2 r_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}}}{\alpha_1 \alpha_2}} \quad (4.4)$$

and arbitrary  $J_1$  and  $J_2$ . Let  $N_L = B_L \cap W^s(\mathcal{Z}_0)$  and  $N_R = B_R \cap W^u(\mathcal{Z}_0)$ . Then,

$$\omega(N_L) = \left\{ \left( \phi_0 + \frac{1}{\alpha_1 + \alpha_2} \ln \frac{\alpha_1 l_1}{\alpha_2 l_2}, 0, 0, (\alpha_1 + \alpha_2)(\alpha_1 l_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}}(\alpha_2 l_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}}, J_1, J_2, 0 \right) \right\},$$

$$\alpha(N_R) = \left\{ \left( \frac{1}{\alpha_1 + \alpha_2} \ln \frac{\alpha_1 r_1}{\alpha_2 r_2}, 0, 0, (\alpha_1 + \alpha_2)(\alpha_1 r_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}}(\alpha_2 r_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}}, J_1, J_2, 1 \right) \right\}$$

for all  $J_1$  and  $J_2$ .

*Proof.* The statement (i) can be verified directly. The statement (ii) is a simple consequence of (i) together with the fact that  $\phi(\xi) \rightarrow \phi^*$ ,  $w(\xi) \rightarrow w^*$ ,  $u(\xi) \rightarrow 0$  and  $v(\xi) \rightarrow 0$  as  $\xi \rightarrow \infty$  for the stable manifold and as  $\xi \rightarrow -\infty$  for the unstable manifold.

For the statement (iii), we present only the proof regarding the intersection of  $W^s(\mathcal{Z}_0)$  and  $B_L$ . Suppose

$$z^0 = (\phi^0, u^0, v^0, w^0, J_1^0, J_2^0, 0) = (\phi_0, u^0, h(0)(\alpha_2 l_2 - \alpha_1 l_1), \alpha_1^2 l_1 + \alpha_2^2 l_2, J_1^0, J_2^0, 0)$$

is a point in  $B_L \cap W^s(\mathcal{Z}_0)$ . Then, using the integrals  $H_1$ ,  $H_2$  and  $H_3$ , the solution  $z(\xi) = (\phi(\xi), u(\xi), v(\xi), w(\xi), J_1^0, J_2^0, 0)$  of system (2.14) with initial condition  $z(0) = z^0$  satisfies

$$\begin{aligned} H_1(z(\xi)) &= w(\xi) - \frac{\alpha_2 - \alpha_1}{h(0)} v(\xi) - \frac{\alpha_1 \alpha_2}{2h^2(0)} u^2(\xi) = A, \\ H_2(z(\xi)) &= \phi(\xi) - \frac{1}{\alpha_2} \ln |\alpha_1 v(\xi)/h(0) + w(\xi)| = B, \\ H_3(z(\xi)) &= \phi(\xi) + \frac{1}{\alpha_1} \ln |\alpha_2 v(\xi)/h(0) - w(\xi)| = C \end{aligned}$$

for some constants  $A$ ,  $B$  and  $C$ , and for all  $\xi$ . From the initial condition, we get

$$B = \phi_0 - \frac{\ln(\alpha_1 + \alpha_2)}{\alpha_2} - \frac{\ln(\alpha_2 l_2)}{\alpha_2} \quad \text{and} \quad C = \phi_0 + \frac{\ln(\alpha_1 + \alpha_2)}{\alpha_1} + \frac{\ln(\alpha_1 l_1)}{\alpha_1}.$$

Since  $u(\xi) \rightarrow 0$  and  $v(\xi) \rightarrow 0$  as  $\xi \rightarrow +\infty$ , we have that  $w(+\infty) = A$  and

$$C - B = \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} \ln w(+\infty) = \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} \ln A.$$

Hence,

$$w(+\infty) = A = (\alpha_1 + \alpha_2)(\alpha_1 l_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 l_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}}.$$

Therefore,

$$u^0 = -[\text{sgn}(v^0)] \sqrt{2} h(0) \sqrt{l_1 + l_2 - \frac{A}{\alpha_1 \alpha_2}} \quad \text{and} \quad \phi(+\infty) = \phi_0 + \frac{1}{\alpha_1 + \alpha_2} \ln \frac{\alpha_1 l_1}{\alpha_2 l_2}.$$

The choice of the sign for  $u^0$  comes from the consideration that the stable eigenvector  $n_-$  in (4.2) has  $u$  and  $v$  components with opposite signs. Thus,  $B_L$  and  $W^s(\mathcal{Z}_0)$  intersect at the points with  $u = u^0$  given above, and all  $J_1$  and  $J_2$ . If  $N_L = B_L \cap W^s(\mathcal{Z}_0)$ , then  $\omega(N_L) = \{(\phi(+\infty), 0, 0, w(+\infty), J_1, J_2, 0)\}$ . The above formulas for  $\phi(+\infty)$  and  $w(+\infty) = A$  gives the desired characterization of  $\omega(N_L)$ . Lastly, since the stable manifold is completely characterized, one can compute its tangent space at each intersection point to verify the transversality of the intersection. It is slightly complicated but straightforward. We will omit the detail here.  $\square$

Part (iii) of this result implies that the boundary layer on the left end will be an orbit of (2.14) from  $(\phi_0, u_L, \alpha_2 l_2 - \alpha_1 l_1, \alpha_1^2 l_1 + \alpha_2^2 l_2, J_1, J_2, 0) \in B_L$  to the point

$$z_L = \left( \phi_0 + \frac{1}{\alpha_1 + \alpha_2} \ln \frac{\alpha_1 l_1}{\alpha_2 l_2}, 0, 0, (\alpha_1 + \alpha_2)(\alpha_1 l_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 l_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}}, J_1, J_2, 0 \right) \in \mathcal{Z}_0,$$

where  $U_L$  is given by the display (4.3) and  $I_1$  and  $I_2$  are arbitrary at this moment; and that on the right end will be a backward orbit of (2.14) from the point  $(0, u_R, \alpha_2 r_2 - \alpha_1 r_1, \alpha_1^2 r_1 + \alpha_2^2 r_2, J_1, J_2, 1) \in B_R$  to the point

$$z_R = \left( \frac{1}{\alpha_1 + \alpha_2} \ln \frac{\alpha_1 r_1}{\alpha_2 r_2}, 0, 0, (\alpha_1 + \alpha_2)(\alpha_1 r_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 r_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}}, J_1, J_2, 1 \right) \in \mathcal{Z}_0,$$

where  $u_R$  is given by the display (4.4) and  $J_1$  and  $J_2$  are arbitrary at this moment. It turns out that there is a unique pair of numbers  $J_1$  and  $J_2$  so that the corresponding points  $z_L$  and  $z_R$  can be connected by a regular layer solution on  $\mathcal{Z}_0$ . The regular orbit together with the two boundary layer orbits provide the singular orbit.

## 4.2 Slow dynamics and regular layers

We now examine the slow flow in the vicinity of the slow manifold  $\mathcal{Z}_0 = \{u = v = 0\}$  for regular layers. Note that system (2.12) resulting from (2.10) by setting  $\mu = 0$  reduces to  $u = v = 0$  and

$$\dot{J}_1 = 0, \quad \dot{J}_2 = 0, \quad \dot{\tau} = 1.$$

The information on  $\phi$  and  $w$  is lost. This indicates that the slow flow in the vicinity of  $\mathcal{Z}_0$  is itself a singular perturbation problem. To see this, we zoom into an  $O(\mu)$ -neighborhood of  $\mathcal{Z}_0$  by blowing up the  $u$  and  $v$  coordinates; that is, we make a scaling  $u = \mu p$  and  $v = \mu q$ . System (2.10) becomes

$$\begin{aligned} \dot{\phi} &= \frac{1}{h(\tau)} p, \quad \mu \dot{p} = q, \quad \mu \dot{q} = pw + \mu \frac{h_\tau(\tau)}{h(\tau)} q + (\alpha_1 J_1 - \alpha_2 J_2), \\ \dot{w} &= \mu \frac{\alpha_1 \alpha_2}{h^2(\tau)} pq + \frac{\alpha_2 - \alpha_1}{h(\tau)} pw - \frac{\alpha_1^2 J_1 + \alpha_2^2 J_2}{h(\tau)}, \\ \dot{J}_1 &= 0, \quad \dot{J}_2 = 0, \quad \dot{\tau} = 1, \end{aligned} \tag{4.5}$$

which is indeed a singular perturbation problem. When  $\mu = 0$ , the system reduces to

$$\begin{aligned} \dot{\phi} &= \frac{1}{h(\tau)} p, \quad 0 = q, \quad 0 = pw + (\alpha_1 J_1 - \alpha_2 J_2), \\ \dot{w} &= \frac{\alpha_2 - \alpha_1}{h(\tau)} pw - \frac{\alpha_1^2 J_1 + \alpha_2^2 J_2}{h(\tau)}, \\ \dot{J}_1 &= 0, \quad \dot{J}_2 = 0, \quad \dot{\tau} = 1. \end{aligned} \tag{4.6}$$

Dynamics of  $\phi$  and  $w$  survives in this limiting process. For this system, the slow manifold is

$$\mathcal{S}_0 = \left\{ p = \frac{\alpha_2 J_2 - \alpha_1 J_1}{w}, q = 0 \right\}.$$

The corresponding fast system is

$$\begin{aligned} \phi' &= \mu \frac{p}{h(\tau)}, \quad p' = q, \quad q' = pw + (\alpha_1 J_1 - \alpha_2 J_2) + \mu \frac{h_\tau(\tau)}{h(\tau)} q, \\ w' &= \mu^2 \frac{\alpha_1 \alpha_2}{h^2(\tau)} pq + \mu \frac{\alpha_2 - \alpha_1}{h(\tau)} pw - \mu \frac{\alpha_1^2 J_1 + \alpha_2^2 J_2}{h(\tau)}, \\ J_1' &= 0, \quad J_2' = 0, \quad \tau' = 0. \end{aligned} \tag{4.7}$$

The limiting system of (4.7) when  $\mu = 0$  is

$$\begin{aligned} \phi' &= 0, \quad p' = q, \quad q' = pw + (\alpha_1 J_1 - \alpha_2 J_2), \\ w' &= 0, \quad J_1' = 0, \quad J_2' = 0, \quad \tau' = 0. \end{aligned} \tag{4.8}$$

The slow manifold  $\mathcal{S}_0$  is the set of equilibria of (4.8). The eigenvalues normal to  $\mathcal{S}_0$  are  $\lambda_\pm(p) = \pm\sqrt{w}$ . In particular, the slow manifold  $\mathcal{S}_0$  is normally hyperbolic, and hence, it persists for system (4.7) for  $\mu > 0$  small (see [6]).

The limiting slow dynamic on  $\mathcal{S}_0$  is governed by system (4.6), which reads

$$\dot{\phi} = \frac{\alpha_2 J_2 - \alpha_1 J_1}{h(\tau)w}, \quad \dot{w} = -\frac{\alpha_1 \alpha_2 (J_1 + J_2)}{h(\tau)}, \quad \dot{J}_i = 0, \quad \dot{\tau} = 1.$$

The general solution is characterized as:  $J_1$  and  $J_2$  are arbitrary constants, and

$$\begin{aligned} \tau(x) &= \tau_0 + x, \quad w(x) = w_0 - \alpha_1 \alpha_2 (J_1 + J_2) \int_0^x \frac{1}{h(\tau_0 + s)} ds, \\ \phi(x) &= \nu_0 + (\alpha_2 J_2 - \alpha_1 J_1) \int_0^x \frac{1}{h(\tau_0 + s)w(s)} ds \\ &= \nu_0 - \frac{\alpha_2 J_2 - \alpha_1 J_1}{\alpha_1 \alpha_2 (J_1 + J_2)} \ln \left| 1 - \frac{\alpha_1 \alpha_2 (J_1 + J_2)}{w_0} \int_0^x \frac{1}{h(\tau_0 + s)} ds \right|. \end{aligned} \tag{4.9}$$

where  $\tau_0 = \tau(0)$ ,  $\phi(0) = \nu_0$  and  $w(0) = w_0$ . Note that, if  $J_1 + J_2 = 0$ , then  $w(x) = w_0$  and

$$\phi(x) = \nu_0 + \frac{\alpha_2 J_2 - \alpha_1 J_1}{w_0} \int_0^x \frac{1}{h(\tau_0 + s)} ds.$$

The latter is the limit of  $\phi(x)$  in (4.9) as  $J_1 + J_2 \rightarrow 0$ . We thus use the unified formula (4.9) even if  $J_1 + J_2 = 0$ .

To identify the slow portion of the singular orbit on  $\mathcal{S}_0$ , we need to examine the  $\omega$ -limit (resp. the  $\alpha$ -limit) set of  $M_L^\mu \cap W^s(\mathcal{S}_0)$  (resp.  $M_R^\mu \cap W^u(\mathcal{S}_0)$ ) as  $\mu \rightarrow 0$ . To do this, we fix an  $O(1)$ -neighborhood of  $\mathcal{S}_0$ . In terms of  $U$  and  $V$ , this neighborhood is of order  $O(\mu)$ . For  $\mu > 0$  small,



the time taken in terms of  $\xi$  for  $M_L^\mu$  and  $M_R^\mu$  to evolve to any  $O(\mu)$ -neighborhood of  $\{u = v = 0\}$  is of order  $O(\mu|\ln \mu|)$ . Thus, the  $\lambda$ -Lemma ([5]) implies that  $M_L^\mu$  (resp.  $M_R^\mu$ ) is  $C^1$   $O(\mu)$ -close to  $M_L^0$  (resp.  $M_R^0$ ) in any  $O(\mu)$ -neighborhood of  $\{u = v = 0\}$ . Therefore, in an  $O(1)$ -neighborhood of  $\mathcal{S}_0$  in terms of  $p$  and  $q$ ,  $M_L^\mu$  (resp.  $M_R^\mu$ ) intersects  $W^s(\mathcal{S}_0)$  (resp.  $W^u(\mathcal{S}_0)$ ) transversally. And, by abusing the notations, if  $N_L = M_L^0 \cap W^s(\mathcal{S}_0)$  and  $N_R = M_R^0 \cap W^u(\mathcal{S}_0)$ , then  $\omega(N_L)$  and  $\alpha(N_R)$  have the same descriptions as those in Proposition 4.1 with  $u = v = 0$  replacing by  $p = (\alpha_2 J_2 - \alpha_1 J_1)/w$  and  $q = 0$ .

The slow orbit should be one given by (4.9) that connects  $\omega(N_L)$  and  $\alpha(N_R)$ . Let  $\bar{M}_L$  (rep.  $\bar{M}_R$ ) be the forward (resp. backward) image of  $\omega(N_L)$  (resp.  $\alpha(N_R)$ ) under the slow flow (4.6).

**Proposition 4.2.**  *$\bar{M}_L$  and  $\bar{M}_R$  intersect transversally along the unique orbit given by (4.9) from  $x = 0$  to  $x = 1$  with*

$$\tau_0 = 0, \quad w_0 = (\alpha_1 + \alpha_2)(\alpha_1 l_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 l_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}}, \quad \nu_0 = \phi_0 + \frac{1}{\alpha_1 + \alpha_2} \ln \frac{\alpha_1 l_1}{\alpha_2 l_2},$$

and  $J_1$  and  $J_2$  are as given in Theorem 2.4.

*Proof.* We show first that  $\bar{M}_L$  and  $\bar{M}_R$  intersect along the orbit with the above characterization. In view of (4.9) and the descriptions for  $\omega(N_L)$  and  $\alpha(N_R)$  in Proposition 4.1, the intersection is uniquely determined by

$$\begin{aligned} \tau_0 &= 0, \quad w(0) = (\alpha_1 + \alpha_2)(\alpha_1 l_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 l_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}}, \\ w(1) &= (\alpha_1 + \alpha_2)(\alpha_1 r_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 r_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}}, \\ \nu_0 &= \phi(0) = \phi_0 + \frac{1}{\alpha_1 + \alpha_2} \ln \frac{\alpha_1 l_1}{\alpha_2 l_2}, \quad \phi(1) = \frac{1}{\alpha_1 + \alpha_2} \ln \frac{\alpha_1 r_1}{\alpha_2 r_2}. \end{aligned}$$

Substituting into (4.9) gives

$$\begin{aligned} J_1 + J_2 &= \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2 \int_0^1 h^{-1}(x) dx} \left( (\alpha_1 l_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 l_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} - (\alpha_1 r_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 r_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} \right), \\ \alpha_2 J_2 - \alpha_1 J_1 &= \frac{(\alpha_1 + \alpha_2) \left( (\alpha_1 l_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 l_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} - (\alpha_1 r_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 r_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} \right)}{\left( \frac{\alpha_2}{\alpha_1 + \alpha_2} \ln \frac{r_1}{l_1} + \frac{\alpha_1}{\alpha_1 + \alpha_2} \ln \frac{r_2}{l_2} \right) \int_0^1 h^{-1}(x) dx} \times \\ &\quad \times \left( \phi_0 + \frac{1}{\alpha_1 + \alpha_2} \ln \frac{l_1 r_2}{l_2 r_1} \right), \end{aligned}$$

which in turn yields the expressions for  $J_1$  and  $J_2$ . To see the transversality of the intersection, it suffices to show that  $\omega(N_L) \cdot 1$  (the image of  $\omega(N_L)$  under the time one map of the flow of

system (4.6)) is transversal to  $\alpha(N_R)$  on  $\mathcal{S}_0 \cap \{\tau = 1\}$ . If we use  $(\phi, w, J_1, J_2)$  as a coordinate system on  $\mathcal{S}_0 \cap \{\tau = 1\}$ , then the set  $\omega(N_L) \cdot 1$  is given by  $\{(\phi(J_1, J_2), w(J_1, J_2), J_1, J_2)\}$  with

$$\begin{aligned}\phi(J_1, J_2) &= \phi_0 + \frac{1}{\alpha_1 + \alpha_2} \ln \frac{\alpha_1 l_1}{\alpha_2 l_2} - \frac{\alpha_2 J_2 - \alpha_1 J_1}{\alpha_1 \alpha_2 (J_1 + J_2)} \ln \left( 1 - \frac{\rho_0 \alpha_1 \alpha_2 (J_1 + J_2)}{w_0} \right), \\ w(J_1, J_2) &= (\alpha_1 + \alpha_2) (\alpha_1 l_1)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} (\alpha_2 l_2)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} - \rho_0 \alpha_1 \alpha_2 (J_1 + J_2),\end{aligned}$$

where  $\rho_0 = \int_0^1 h^{-1}(x) dx$ . Thus, the tangent space to  $\omega(N_L) \cdot 1$  restricted on  $\mathcal{S}_0 \cap \{\tau = 1\}$  is spanned by  $(\phi_{J_1}, w_{J_1}, 1, 0) = (\phi_{J_1}, -\rho_0 \alpha \beta, 1, 0)$  and  $(\phi_{J_2}, w_{J_2}, 0, 1) = (\phi_{J_2}, -\rho_0 \alpha \beta, 0, 1)$ . In view of the display in Proposition 4.1, the tangent space to  $\alpha(N_R)$  restricted on  $\mathcal{S}_0 \cap \{\tau = 1\}$  is spanned by  $(0, 0, 1, 0)$  and  $(0, 0, 0, 1)$ . Note that  $\mathcal{S}_0 \cap \{\tau = 1\}$  is four dimensional. Thus, it suffices to show that the above four vectors are linearly independent, or equivalently,  $\phi_{J_1} \neq \phi_{J_2}$ . The latter can be verified by a direct computation. Indeed, if  $J_1 + J_2 \neq 0$  at the intersection points, then

$$\phi_{J_1} - \phi_{J_2} = \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2 (J_1 + J_2)} \ln \left( 1 - \frac{\rho_0 \alpha_1 \alpha_2 (J_1 + J_2)}{w_0} \right) \neq 0;$$

if  $J_1 + J_2 = 0$  at intersection points, then  $\phi(J_1, J_2) = \nu_0 + \rho_0(\alpha_2 J_2 - \alpha_1 J_1)/w_0$  and hence  $\phi_{J_1} - \phi_{J_2} = -\rho_0(\alpha_1 + \alpha_2)/w_0 \neq 0$ .  $\square$

### 4.3 Proof of Theorem 2.4

We provide a detailed version of Theorem 2.4 and its proof.

**Theorem 4.3.** *Assume that  $\alpha_1 l_1 \neq \alpha_2 l_2$  and  $\alpha_1 r_1 \neq \alpha_2 r_2$ . For  $\mu > 0$  small, the connecting problem (2.10) and (4.1) has a unique solution near a singular orbit. The singular orbit is the union of two fast orbits of system (2.14) and one slow orbit of system (4.6); more precisely, with both  $J_1$  and  $J_2$  given in Theorem 2.4,*

- (i) *the fast orbit representing the limiting boundary layer at  $x = 0$  lies on  $B_L \cap W^s(\mathcal{Z}_0)$  from  $B_L$  to  $\omega(N_L) \subset \mathcal{Z}_0$  whose starting point has the  $u$ -component given by (4.3) in Propositions 4.1,*
- (ii) *the fast orbit representing the limiting boundary layer at  $x = 1$  lies on  $B_R \cap W^u(\mathcal{Z}_0)$  from  $B_R$  to  $\alpha(N_R) \subset \mathcal{Z}_0$  whose starting point has the  $u$ -component given by (4.4) in Propositions 4.1,*
- (iii) *the slow orbit on  $\mathcal{S}_0$  connecting the two boundary layers from  $x = 0$  to  $x = 1$  is displayed in (4.9) together with the quantities in Proposition 4.2.*

*Proof.* The singular orbit has been studied in Sections 4.1 and 4.2, which is summarized in (i), (ii) and (iii) of this theorem. It remains to show the existence and uniqueness of a solution near the singular orbit for  $\mu > 0$ . Recall that  $M_L^\mu$  (resp.,  $M_R^\mu$ ) is the union of all forward (resp., backward) orbits starting from  $B_L$  (resp.,  $B_R$ ). It suffices to show that, for  $\mu > 0$  small,  $M_L^\mu$  and  $M_R^\mu$  intersect

transversally with each other around the singular orbit. We note that the assumption  $\alpha_1 l_1 \neq \alpha_2 l_2$  and  $\alpha_1 r_1 \neq \alpha_2 r_2$  imply the vector field of (2.10) is not tangent to  $B_L$  and  $B_R$ , and hence,  $M_L^\mu$  and  $M_R^\mu$  are smooth invariant manifolds.

For  $\mu > 0$  small, the evolutions of  $M_L^\mu$  and  $M_R^\mu$  from  $B_L$  and  $B_R$ , respectively, to an  $\mu$ -neighborhood of  $\mathcal{Z}_0$  along the two boundary layers are governed by system (2.13). Since, for system (2.14),  $M_L^0$  and  $M_R^0$  intersect  $W^s(\mathcal{Z}_0)$  and  $W^u(\mathcal{Z}_0)$  transversally, we have that  $M_L^\mu$  and  $M_R^\mu$  intersect  $W^s(\mathcal{Z}_0)$  and  $W^u(\mathcal{Z}_0)$  transversally. As discussed in Section 4.2, in terms of the blow-up coordinates,  $M_L^\mu$  and  $M_R^\mu$  intersect  $W^s(\mathcal{S}_0)$  and  $W^u(\mathcal{S}_0)$  transversally for system (4.7). And, if we denote  $N_L = M_L^0 \cap W^s(\mathcal{S}_0)$  and  $N_R = M_R^0 \cap W^u(\mathcal{S}_0)$ , then the vector field on  $\mathcal{S}_0$  is not tangent to  $\omega(N_L)$  and  $\alpha(N_R)$ . Furthermore, the traces  $\bar{M}_L$  and  $\bar{M}_R$  of  $\omega(N_L)$  and  $\alpha(N_R)$  respectively under the slow flow on  $\mathcal{S}_0$  intersect transversally. All conditions for the Exchange Lemma (see [30] and also [20, 18, 19]) are satisfied, and hence,  $M_L^\mu$  and  $M_R^\mu$  intersect transversally. The intersection has dimension

$$\dim M_L^\mu + \dim M_R^\mu - 7 = 4 + 4 - 7 = 1,$$

which is the orbit of the unique solution for the connecting problem near the singular orbit.  $\square$

*Remark 4.1.* We have considered the situation that  $\alpha_1 l_1 \neq \alpha_2 l_2$  and  $\alpha_1 r_1 \neq \alpha_2 r_2$ . In case that  $\alpha_1 l_1 = \alpha_2 l_2$  or  $\alpha_1 r_1 = \alpha_2 r_2$ , then  $B_L$  or  $B_R$  are on the slow manifold  $\mathcal{S}_0$  and hence there is no boundary layer at  $x = 0$  or  $x = 1$ .

#### 4.4 A Special Case

We conclude the paper by examining a special case. Consider the one-dimensional limit PNP system (1.1) with the special boundary conditions

$$\phi(0) = \phi_0, \phi(1) = 0, \alpha_1 c_1(0) = \alpha_1 c_1(1) = \alpha_2 c_2(0) = \alpha_2 c_2(1) = k > 0. \quad (4.10)$$

One sees that  $(c_1^0(x), c_2^0(x), \phi^0(x)) = (k/\alpha_1, k/\alpha_2, (1-x)\phi_0)$  is a steady-state solution. Motivated by many works on Lyapunov functions for systems of PNP-types (see, e.g., [3]), we set

$$L(t) = \sum_{j=1}^2 \frac{1}{D_j} \int_0^1 h(x) (c_j(x, t) - c_j^0(x)) \ln \frac{c_j(x, t)}{c_j^0(x)}.$$

It turns out that  $L(t)$  is a Lyapunov functional. In fact, using the equation and integration by parts,

$$\begin{aligned} L'(t) &= \sum_{j=1}^2 \frac{1}{D_j} \int_0^1 h(x) \partial_t c_j(x, t) \left( \ln \frac{c_j(x, t)}{c_j^0(x)} + \frac{(c_j(x, t) - c_j^0(x))}{c_j(x, t)} \right) \\ &= - \int_0^1 h \left( \frac{c_1 + c_1^0}{c_1^2} (\partial_x c_1)^2 + \frac{c_2 + c_2^0}{c_2^2} (\partial_x c_2)^2 \right) \\ &\quad - \lambda \int_0^1 h (\alpha_1 c_1 - \alpha_2 c_2)^2 - \lambda k \int_0^1 h (\alpha_1 c_1 - \alpha_2 c_2) (\ln(\alpha_1 c_1) - \ln(\alpha_2 c_2)) \leq 0. \end{aligned}$$

Also,  $L'(t) = 0$  if and only if  $\partial_x c_1 = \partial_x c_2 = \alpha_1 c_1 - \alpha_2 c_2 = 0$ , which in turn imply that  $c_1 = c_1^0$ ,  $c_2 = c_2^0$  and  $\phi = \phi^0$ . Due to the invariant principle (Proposition 2.2 for one-dimensional case), one can check that  $L(t)$  is equivalent to

$$\sum_{i=1}^2 \int_0^1 (c_i(x, t) - c_i^0(x))^2 dx$$

if  $c_i(x, 0) > 0$  for  $x \in [0, 1]$ , and hence,  $c_i \rightarrow c_i^0$  in  $L^2(0, 1)$  exponentially as  $t \rightarrow \infty$ . This shows a significant difference in asymptotic behavior between the total non-flux boundary conditions (see [3]) and the boundary conditions (3.4) considered in this work for the PNP systems.

## References

- [1] V. Barcion, D.-P. Chen, and R. S. Eisenberg, Ion flow through narrow membrane channels: Part II. *SIAM J. Appl. Math.* **52** (1992), 1405-1425.
- [2] V. Barcion, D.-P. Chen, R. S. Eisenberg, and J. W. Jerome, Qualitative properties of steady-state Poisson-Nernst-Planck systems: Perturbation and simulation study. *SIAM J. Appl. Math.* **57** (1997), 631-648.
- [3] P. Biler and J. Dolbeault, Long time behavior of solutions to Nernst-Planck and Debye-Hückel drift-diffusion systems. *Ann. Henri Poincaré* **1** (2000), 461-472.
- [4] P. Biler, W. Hebisch, and T. Nadzieja, The Debye system: existence and large time behavior of solutions. *Nonlinear Analysis TMA* **23** (1994), 1189-1209.
- [5] B. Deng, The Sil'niko problem, exponential expansion, strong  $\lambda$ -lemma,  $C^1$  linearization and homoclinic bifurcation. *J. Differential Equations* **79**(1989), 189-231.
- [6] N. Fenichel, Persistence and smoothness of invariant manifolds for flows. *Indiana Univ. Math. J.* **21**(1971), 193-226.

- [7] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations. *J. Differential Equations* **31**(1979), 53–98.
- [8] H. Gajewski and K. Gröger, On the basic equations for carrier transport in semiconductors. *J. Math. Anal. Appl.* **113**(1989), 12–35.
- [9] H. Gajewski and K. Gröger, Semiconductor equations for variable mobilities based on Boltzmann statistics or Fermi-Dirac statistics. *Math. Nachr.* **140**(1989), 7–36.
- [10] K. Gröger, Initial boundary value problems from semiconductor device theory. *Math. Nachr.* **129**(1986), 167–174.
- [11] K. Gröger, Initial boundary value problems from semiconductor device theory. *Z. Angew. Math. Mech.* **67**(1987), 345–355.
- [12] J.K. Hale and G. Raugel, A Damped Hyperbolic Equation on Thin Domains, *Trans. Amer. Math. Soc.* **329**(1992), 185–219.
- [13] J.K. Hale and G. Raugel, Reaction-Diffusion Equation on Thin Domains. *J. Math. Pures et Appl.* **71**(1992), 33–95.
- [14] M. Hirsch, C. Pugh, and M. Shub, *Invariant Manifolds*. Lect. Notes in Math. **583**, Springer-Verlag, New York, 1976.
- [15] M. Holmes, Nonlinear Ionic Diffusion Through Charged Polymeric Gels. *SIAM J. Appl. Math.* **50**(1990), 839–852.
- [16] J.W. Jerome, Consistency of Semiconductor Modeling: An Existence/Stability Analysis for the Stationary Van Roosbroeck System. *SIAM J. Appl. Math.* **45**(1985), 565–590.
- [17] J. W. Jerome and T. Kerkhoven, A finite element approximation theory for the drift-diffusion semiconductor model. *SIAM J. Numer. Anal.* **28** (1991), 403–422.
- [18] C. Jones, Geometric singular perturbation theory. *Dynamical systems (Montecatini Terme, 1994)*, 44–118. Lect. Notes in Math. **1609**, Springer, Berlin, 1995.
- [19] C. Jones, T. Kaper, and N. Kopell, Tracking invariant manifolds up to exponentially small errors. *SIAM J. Math. Anal.* **27**(1996), 558–577.
- [20] C. Jones and N. Kopell, Tracking invariant manifolds with differential forms in singularly perturbed systems. *J. Differential Equations* **108**(1994), 64–88.
- [21] J. Keener and J. Sneyd, *Mathematical Physiology*. Interdisciplinary Applied Mathematics, Springer-Verlag, New York

- [22] W. Liu, Exchange Lemmas for Singular Perturbation Problems with Certain Turning Points. *J. Differential Equations* **167**(2000), 134-180.
- [23] W. Liu, Geometric singular perturbation approach to steady-state Poisson-Nernst-Planck systems. *SIAM J. Appl. Math.* (to appear).
- [24] M. S. Mock, Asymptotic behavior of solutions of transport equations for semiconductor devices. *J. Math. Anal. Appl.* **49** (1975), 215–225.
- [25] J.-K. Park and J. W. Jerome, Qualitative properties of steady-state Poisson-Nernst-Planck systems: Mathematical study. *SIAM J. Appl. Math.* **57** (1997), 609-630.
- [26] G. Raugel and G. R. Sell, Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions. *J. Amer. Math. Soc.* **6** (1993), 503–568.
- [27] I. Rubinstein, *Electro-Diffusion of Ions*. SIAM Studies in Applied Mathematics, SIAM, Philadelphia, PA, 1990.
- [28] T. Seidman, Time-dependent solutions of a nonlinear system arising in semiconductor theory—II, boundedness and periodicity. *Nonlinear Analysis TMA* **10**(1986), 491-502.
- [29] R. Temam and M. Ziane, Navier-Stokes equations in three-dimensional thin domains with various boundary conditions. *Adv. Differential Equations* **1** (1996), 499–546.
- [30] S.-K. Tin, N. Kopell, and C. Jones, Invariant manifolds and singularly perturbed boundary value problems. *SIAM J. Numer. Anal.* **31**(1994), 1558-1576.